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By this mail I would send you a copy of my "Theory of Equations," which I wish to present to your Library. Having a certain number of copies left over, I have thought the work may prove of interest from being, as far as I have been able to ascertain, the first of its kind — reprints excepted, published in the United States.

Yours truly
John Macnie

# A TREATISE

ON

# THE THEORY AND SOLUTION

OF

# ALGEBRAICAL EQUATIONS.

BY

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A. S. BARNES & COMPANY,

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# PREFACE.

THE following treatise is designed to present, as succinctly as is consistent with a due presentation of the subject, the general theory of algebraical equations, while special attention is given to the analysis and solution of equations with numerical coefficients.

The work may be regarded as a complement to the more advanced treatises on algebra, in which the general theory of equations, if discussed at all, is necessarily compressed into a space altogether inadequate to a satisfactory exposition of this important branch of mathematical study. In order to preserve this character of the work as a sequel to ordinary algebra, algebraical methods have been carefully adhered to throughout, except in a few articles, in which trigonometrical expressions have unavoidably been introduced. The treatise may therefore be easily read by those who have passed through the usual course in algebra.

The work had its first inception in some vain attempts at so modifying Sturm's Method as to lessen the great labor attending an analysis by that method of most equations of above the fourth degree. Becoming convinced that, from the nature of the case, no such modification is possible that does

not impair the certainty which constitutes the chief excellency of the method, the author turned his attention to an investigation of the possibility of a satisfactory analysis by means of Fourier's Theorem. This he soon saw could be effected if, by any means, the presence of imaginary roots in a given interval could be readily ascertained. The results of this investigation are given (Chap. X) in a method of analysis based upon Fourier's Theorem, a method that possesses the merit of at least great facility as compared with that of Sturm.

The method involves, however, an extension of the application of Horner's Method, especially a generalization of the principle of trial divisors, not given in the great majority of treatises upon equations. This first suggested the idea of writing a short account of Horner's Method in which its capabilities should be fully exhibited, with the method of analysis based on that method in conjunction with Fourier's Theorem. A natural extension of that idea led to the present treatise, the first upon this subject that, as far as the author has been able to ascertain, has been published in the United States.

The treatise, though kept within its present dimensions by the exclusion of much that had been prepared, will be found to contain all the propositions generally given in an elementary treatise upon this subject, with a few exceptions. Thus Newton's method of approximation, and that of Lagrange, have been excluded, as being entirely superseded by that of Horner, to which, even in the most favorable cases, they are inferior in symmetry, compactness, and facility. The theory of determinants has not been introduced, on the ground that a suitable account would require to itself a volume of the dimensions of the present treatise. The student desirous of information upon the foregoing and other omitted topics may

consult Todhunter's excellent treatise, the best, upon the whole, with which the author is acquainted.

On account of the interest naturally attaching to the subject, the algebraical solution of equations has (Chap. VII) been treated of at some length. In Art. 121, 133–146, are given some results that may be found of interest, and which, it is believed, are now published for the first time.

A chapter (VIII) has been devoted to Sturm's Theorem, thus enabling the student to form for himself a clear estimate of the peculiar excellencies and disadvantages of the method of analysis based upon that theorem, and to institute a fair comparison between it and the method explained in Chap. X.

A short chapter on cubic equations has been inserted, in which a method of procedure is given that relieves the solution of the cubic from much of its tentative character, and reduces the arithmetical labor to a minimum.

To render the treatise more convenient for the work of the class-room, the subject-matter has been thrown, as far as possible, into the form of propositions with their dependent corollaries. The number of exercises given will, it is hoped, be found sufficient in number to illustrate every part of the subject and not so difficult as to needlessly consume time.

JOHN MACNIE.

NEWBURGH, *August*, 1875.

# CONTENTS.

## INTRODUCTION.



## CHAPTER I.

### FUNDAMENTAL PROPERTIES OF EQUATIONS.



## CHAPTER II.

### IMAGINARY EXPRESSIONS. CAUCHY'S THEOREM.

## CHAPTER III.

### GENERAL PROPERTIES OF EQUATIONS

*dependent on the principle that every equation has a root.*



## CHAPTER IV.

### TRANSFORMATION OF EQUATIONS.



## CHAPTER V.

### LIMITS OF THE ROOTS OF EQUATIONS.

## CHAPTER VI.

### DEPRESSION OF EQUATIONS.

## CHAPTER VII.

### SOLUTION OF EQUATIONS BY GENERAL FORMULAS.



## CHAPTER VIII.

### STURM'S THEOREM.



## CHAPTER IX.

### HORNER'S METHOD.



## CHAPTER X.

### ANALYSIS OF EQUATIONS BY FOURIER'S THEOREM.

## CHAPTER XI.

### CUBIC EQUATIONS.



## CHAPTER XII.

### SYMMETRICAL FUNCTIONS OF THE ROOTS.



## CHAPTER XIII.

### ELIMINATION.

# THEORY AND SOLUTION

OF

# ALGEBRAICAL EQUATIONS.

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## INTRODUCTION.

**1.** Algebraical equations of the first and second degrees are generally fully treated of in elementary works on algebra; the student may therefore be supposed to have a knowledge of the subject so far, and to be acquainted with the meaning of the terms, *equation, root, member*, &c. The present treatise will be devoted to a discussion of the general theory of algebraical equations, and the methods of solution for equations of the third and higher degrees, sometimes called the Higher Equations.

We shall generally express an equation of the $n^{th}$ degree under the form,

$$C_n x^n + C_{n-1} x^{n-1} + \quad . \quad . \quad . \quad . \quad C_2 x^2 + C_1 x + C_0 = 0,$$

in which $C_r$, the coefficient of the $r^{th}$ power of $x$, is a known quantity, positive, negative, or zero. The final term, $C_0$, which may be regarded as the coefficient of $x^0$, is frequently called the absolute or independent term.

The equation is said to be complete when it contains all the powers of $x$ from $x^n$ to $x^0$, otherwise it is incomplete.

When $n$, the exponent of the highest power of $x$, is a positive integer, the equation is said to be rational and integral, that is, $x$ does not occur with fractional or negative exponents.

Thus, $5x^6 + 17x^5 - 39x^3 + 106x^2 - 53x - 17 = 0$ is a rational integral equation; while

$$x^5 - 23x^{\frac{7}{2}} + 5x^3 + 6x^{\frac{2}{3}} - 7 = 0,$$

or

$$x^5 - 23\sqrt{x^7} + 5x^3 + 6\sqrt[3]{x^2} - 7 = 0,$$

is an irrational equation; and

$$7x^4 - 3x^{\frac{3}{2}} + 9x^{-2} + 13x^{-4} + 8 = 0,$$

is both irrational and fractional.

Since irrational and fractional equations may always be reduced to a rational and integral form (see Art. 57), the equations treated of in this work will always be supposed to be rational and integral.

By dividing both members of an equation by the coefficient of the highest power of $x$, we obtain it under the form,

$$x^n + C_{n-1}x^{n-1} + \ldots\ldots C_2x^2 + C_1x + C_0 = 0,$$

a form which will sometimes be found advantageous when enunciating some of the properties of equations.

**2.** Any quantity, real or imaginary, which, when substituted for $x$ in the expression,

$$C_nx^n + C_{n-1}x^{n-1} + \ldots\ldots C_2x^2 + C_1x + C_0,$$

causes the whole to become zero, is said to be a root of the equation,

$$C_nx^n + C_{n-1}x^{n-1} + \ldots\ldots C_2x^2 + C_1x + C_0 = 0.$$

The solution of an equation consists in the determination of these roots.

We know that for the quadratic equation,

$$C_2x^2 + C_1x + C_0 = 0,$$

the roots are expressed by the formula,

$$x = \tfrac{1}{2}(-C_1 \pm \sqrt{C_1^2 - 4C_2C_0}).$$

This formula contains the *algebraical solution* of the equation of the second degree, in terms of the general symbols $C_2$, $C_1$, $C_0$. The *numerical solution* of any given equation of that degree we may obtain by substituting in the formula

the known values of the coefficients. To obtain for the higher equations similar formulas, expressing the roots in terms of the coefficients, was long considered the most important problem in algebra. But, as will hereafter be seen, no such formula has been found for equations beyond the fourth degree, and even those that have been obtained for the cubic and biquadratic are not always available for purposes of numerical computation.

Yet, though the search for general formulas, to express the roots of equations of the fifth and higher degrees, has been entirely unsuccessful, we are in possession of methods by which we can obtain exactly, when commensurable, or approximate to when incommensurable, any root of a numerical equation, that is, of an equation having its coefficients known numbers. These methods depend on the general properties of equations, which we shall first demonstrate as a necessary preliminary to the study of methods of numerical solution.

**3.** An algebraical expression involving $x$ is often conveniently referred to as a function of $x$, which may be defined as follows:

*A Function of* $x$ is an expression that depends for its value on $x$.

Thus, $x^2$, $\sqrt[3]{x}$, $ax^2 + bx + c$, $5x^4 - 7x + 32$, are functions of $x$, since their values depend on that of $x$.

The symbol for a function is one of the letters $f$, $F$, $\phi$, &c., prefixed to the symbol of the variable quantity. Thus, $f(x)$, $F(x)$, $f_1(x)$, $\phi(x)$, $f(y)$, $f(x+y)$, &c., may be used to represent any expression involving $x$ or $y$, or both $x$ and $y$, it being of course understood that, in the same investigation, a particular symbol, $f(x)$ for example, shall always denote the same expression: thus, if

$$\begin{aligned}
f(x) &= 5x^4 - 3x^3 + 2x^2 + 1, \text{ then} \\
f(a) &= 5a^4 - 3a^3 + 2a^2 + 1, \\
f(0) &= 0 - 0 + 0 + 1 = 1, \\
f(1) &= 5 - 3 + 2 + 1 = 5, \\
f(-2) &= 5 \times (-2)^4 - 3 \times (-2)^3 + 2 \times (-2)^2 + 1 = 113, \\
f(3) &= 5 \times 3^4 - 3 \times 3^3 + 2 \times 3^2 + 1 = 343,
\end{aligned}$$

that is, if $f(x)$ denote a certain expression involving $x$, then $f(a)$, $f(0)$, $f(3)$, &c., denote the same expression, or its numerical value, when $a$, 0, 3, &c., are substituted for $x$.

**4.** The symbols, $f_1(x)$, $f_2(x)$, . . . $f_4(x)$, &c., are often employed to denote functions derived from a primitive function, $f(x)$, according to some given law: thus, if

$$\begin{aligned} f(x) &= C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0, \text{ then} \\ f_1(x) &= 4C_4x^3 + 3C_3x^2 + 2C_2x + C_1, \\ f_2(x) &= 4\cdot3\ C_4x^2 + 3\cdot2\cdot C_3x + 2C_2, \\ f_3(x) &= 4\cdot3\cdot2\cdot C_4x + 3\cdot2\cdot C_3, \\ f_4(x) &= 4\ 3\cdot2\cdot C_4. \end{aligned}$$

In this example the law of derivation is that each function, after the first, is derived from that preceding, by multiplying each term by the exponent of $x$ in that term, and then diminishing that exponent by unity. When derived functions are mentioned, it is generally functions derived from a primitive function, $f(x)$, according to the foregoing law, that are meant. The function denoted by $f_1(x)$ is called the *first derived function* of $f(x)$; $f_2(x)$, again, which is the first derived function of $f_1(x)$, is the *second derived function* of $f(x)$, and so on, $f_r(x)$ denoting the $r^{th}$ *derived function* of $f(x)$. When $f(x)$ is of $n$ dimensions, there may evidently be $n$ derived functions. The $n^{th}$ derived function $f_n(x)$, though independent of $x$, may for convenience be included among the derived functions, as being derived from $f(x)$ by a repetition of the process by which the preceding functions were obtained.

EXERCISES.

Write out the derived functions of each of the following functions; and find the values of $f(0)$, $f(1)$, $f(2)$, $f(3)$.

1. $f(x) = x^4 + 7x^3 + 8x^2 - 10x - 13.$
2. $f(x) = 3x^7 + 13x^4 - 8x^3 + 18.$
3. $f(x) = x^5 - 13x^4 + 29x^3 - 52x^2 - 16.$
4. $f(x) = x^8 - 8x^7 + 7x^6 - 6x^5 - 5x^2 + 4.$
5. Find the $r^{th}$ derived function of

$$C_nx^n + C_{n-1}x^{n-1} + \ .\ .\ .\ .\ C_2x^2 + C_1x + C_0.$$

# CHAPTER I.

## FUNDAMENTAL PROPERTIES OF EQUATIONS, INDEPENDENT OF THE PRINCIPLE THAT EVERY EQUATION HAS A ROOT.

**5.** The most fundamental proposition respecting equations is that proved by Cauchy's Theorem, Chap. II, namely, that every equation has a root. Some important propositions may, however, be proved independently of that theorem; and others, again, are necessary as an introduction to it. These form the subject of the present chapter.

**6.** PROP. I.—*If* $f(x) = C_n x^n + C_{n-1} x^{n-1} + \dots C_2 x^2 + C_1 x + C_0$, *be a rational integral function of* $x$, *then any term* $C_r x^r$, *that occurs, may be made to contain the sum of all the terms that follow it, as often as we please, by taking* $x$ *large enough.*

Let $k$ be the numerical value of the greatest coefficient that follows $C_r$, the sum of all the terms, $C_{r-1} x^{r-1} + C_{r-2} x^{r-2} + \dots C_1 x + C_0$, that follow $C_r x^r$, cannot be greater than

$$k(x^{r-1} + x^{r-2} + \dots x + 1)$$

that is, than $k \cdot \dfrac{x^r - 1}{x - 1}$. The quotient of $C_r x^r$ by this is $\dfrac{C_r(x-1)x^r}{k(x^r-1)} = \dfrac{C_r(x-1)}{k(1-x^{-r})}$. The numerator of this expression may be made as great as we please, and the denominator as near to $k$ as we please, by taking $x$ large enough.

COR. 1.—By putting $x = \dfrac{1}{y}$, the function becomes

$$y^{-n}(C_n + C_{n-1} y + \dots C_1 y^{n-1} + C_0 y^n),$$

in which any term may, as above shown, be made to contain, as often as we please, the sum of all the terms that *precede* it, by taking $y$ great enough, that is, by taking $x$ *small* enough.

COR. 2.—When $x$ is large, the term in $f(x)$ containing the highest power of $x$ is the most important, and *vice versa.*

Ex.—If $f(x) = 3x^4 - 7x^3 - 5x^2 + 8x + 11$,
then $f(10) = 3\times 10^4 - 7\times 10^3 - 5\times 10^2 + 8\times 10 + 11$,

in which the first term greatly exceeds the sum of all the remaining terms in value; and in

$$f(\tfrac{1}{10}) = 3(\tfrac{1}{10})^4 - 7(\tfrac{1}{10})^3 - 5(\tfrac{1}{10})^2 + 8(\tfrac{1}{10}) + 11,$$

the last term exceeds the sum of all the preceding terms.

**7.** PROP. II.—*To determine the form that $f(x)$ assumes when $x + y$ is put for $x$, that is, when $x$ is increased or diminished by any quantity $y$.* Let

$$f(x) = C_n x^n + C_{n-1}x^{n-1} + \ldots C_2x^2 + C_1x + C_0; \text{ then}$$
$$f(x+y) = C_n(x+y)^n + C_{n-1}(x+y)^{n-1} + \ldots C_2(x+y)^2 + C_1(x+y) + C_0.$$

Expanding the different powers of $x + y$ in this series, by means of the Binomial Theorem, and arranging the result according to ascending powers of $y$, we obtain,

$$\begin{aligned} f(x+y) = {} & C_n x^n + C_{n-1}x^{n-1} + \ldots\ldots C_2x^2 + C_1x + C_0 \\ & + y\left\{nC_nx^{n-1} + (n-1)C_{n-1}x^{n-2} + (n-2)C_{n-2}x^{n-3} + \ldots\ldots C_1\right\} \\ & + \frac{y^2}{\lfloor 2}\left\{n(n-1)C_nx^{n-2} + (n-1)(n-2)C_{n-1}x^{n-3} + \ldots\ldots 2C_2\right\} \\ & + \frac{y^3}{\lfloor 3}\left\{n(n-1)(n-2)C_nx^{n-3} + (n-1)(n-2)(n-3)C_{n-1}x^{n-4} + \ldots 3\cdot 2\cdot C_3\right\} \\ & + \ldots\ldots\ldots\ldots\ldots\ldots \\ & + \frac{y^r}{\lfloor r}\left\{n(n-1)\ldots(n-r+1)C_nx^{n-r} + (n-1)(n-2)\ldots(n-r)C_{n-1}x^{n-r-1} + \ldots \lfloor r\, C_r\right\} \\ & + \ldots\ldots\ldots\ldots\ldots\ldots \\ & + \frac{y^n}{\lfloor n}\left\{\lfloor n\, C_n\right\}. \end{aligned}$$

Thus we see that the development of $f(x+y)$ is a series of the form,

$$P + Qy + R\frac{y^2}{\lfloor 2} + S\frac{y^3}{\lfloor 3} + \ldots . C_n y^n,$$

where $P$, $Q$, $R$, &c., represent the compound coefficients above, and are all functions of $x$.

The first of the coefficients $P$, is evidently the original function $f(x)$. The second coefficient $Q$, is derived from $P$ by multiplying each term in $P$ by the exponent of $x$ in that term, and then diminishing the exponent by unity, that is (Art. 4), $Q$ is the first derived function of $f(x)$, which we denote by $f_1(x)$. $R$, in like manner, is the second derived function $f_2(x)$, $S$ is the third derived function $f_3(x)$, and so on, $f_r(x)$ being the coefficient of $\dfrac{y^r}{\lfloor r}$.

Employing this notation, we have,

$$f(x+y) = f(x) + f_1(x)y + f_2(x)\frac{y^2}{\lfloor 2} + \ldots f_r(x)\frac{y^r}{\lfloor r} + \ldots f_n(x)\frac{y^n}{\lfloor n},$$

where $\dfrac{1}{\lfloor n} f_n(x)$ is evidently $C_n$.

By the student acquainted with the Differential Calculus, this expression for $f(x+y)$ will be recognized as a particular application of Taylor's Theorem.

**8.** Prop. III.—*If in the function,*

$$f(x) = C_n x^n + C_{n-1} x^{n-1} + \ldots . C_2 x^2 + C_1 x + C_0,$$

*$x$ vary continuously from $a$ to $b$, then will $f(x)$ vary continuously from $f(a)$ to $f(b)$.*

For let $c$ be any value given to $x$ between $a$ and $b$, then $f(c)$ will be the corresponding value of $f(x)$. Let $c + h$ be another value of $x$, then, Prop. II,

$$f(c+h) = f(c) + f_1(c)h + f_2(c)\frac{h^2}{\lfloor n} + \ldots f_{n-1}(c)\frac{h^{n-1}}{\lfloor n-1} + f_n(c)\frac{h^n}{\lfloor n}.$$

Now, Prop. I, Cor. 1, by taking $h$ small enough, the first term of the above series, after $f(c)$, that does not vanish, may

be made to contain the terms that follow it, as often as we please, and this term itself may be made as small as we please; that is, $f(c+h)$ may be made to differ from $f(c)$ by a quantity as small as we please.

Hence, if in a function $f(x)$, $x$ be supposed to pass in succession through all the values from $a$ to $b$, the function will also pass through all the values from $f(a)$ to $f(b)$, seeing that whatever intermediate value $f(c)$ it may have, we may find another as near to it as we please by making a suitable change in the value of $x$. For example,

$$\begin{array}{llllll} \text{Let} & f(x) = x^3 - 18x^2 & + 101x & - 180, & & \\ \text{then} & f(1) = 1 - 18 & + 101 & - 180 & = & -96, \\ & f(10) = 10^3 - 18\times 10^2 & + 101\times 10 & - 180 & = & 30. \end{array}$$

Here we see that for $x = 1$, the value of the function is $-96$; for $x = 10$, the value is 30. What the proposition asserts is that the function may be made to pass through all the values between $-96$ and 30, by giving to $x$ all the values between 1 and 10. It is not asserted that the function is always increasing in this interval. In point of fact, this particular function will, on trial, be found to be sometimes increasing, and sometimes decreasing, between the limits $-96$ and 30.

**9.** Prop. IV.—*If in the equation,*

$$f(x) = C_n n^n + C_{n-1} x^{n-1} + \ldots . C_2 x^2 + C_1 x + C_0 = 0,$$

*two numbers, separately substituted for $x$, give results with contrary signs, one root, at least, of the equation must lie between these numbers.*

Let $a$ and $b$ denote the numbers. Then, since, by supposition, $f(a)$ and $f(b)$ have contrary signs, and, Prop. III, as $x$ increases continuously from $a$ to $b$, $f(x)$ must pass continuously through all the values from $f(a)$ to $f(b)$, one of these values must be zero, as a series of finite quantities cannot pass from one sign to another, without passing through zero.

**10.** In the example given in Art. 8, we found $f(1) = -96$, $f(10) = 30$. By the present theorem, there must be, therefore, at least one root of the equation $x^3 - 18x^2 + 101x - 180$

$= 0$, between 1 and 10; it may be readily ascertained that there are three. Again, though for $f(1)$ we obtain $-96$, and for $f(6)$, the value $-6$, we must not conclude that there is no root of the equation between 1 and 6; there are in fact two.

**11.** PROP. V.—*Every equation of an odd degree has, at least, one real root, of a sign contrary to that of the final term.*

In $f(x) = C_n x^n + C_{n-1} x^{n-1} + \ldots C_2 x^2 + C_1 x + C_0 = 0$, suppose $n$ to be an odd number.

When we put $x = 0$, the function reduces to the final term $C_0$, and has the sign of that term. By taking $x$ large enough, we may, Prop. I, make the first term $C_n x^n$ greater than the sum of all the remaining terms, and the function will have the same sign as $x$, which sign may be taken contrary to that of $C_0$.

Since, then, $f(0)$ has the sign of $C_0$, and $f(a)$, by taking $a$ large enough, and of sign contrary to that of $C_0$, has the contrary sign, there must, Prop. IV, be a real root of $f(x) = 0$ between 0, and $a$, that is, a root of sign contrary to that of the final term $C_0$.

Ex. 1. $7x^5 + 23x^4 - 16x^2 + 8x + 19 = 0,$

must have, at least, one negative root.

Ex. 2. $x^7 + 17x^4 - 105x^3 + 27x - 301 = 0,$

must have, at least, one positive root.

**12.** PROP. VI.—*Every equation of an even degree, having the final term negative, has, at least, two real roots, one of each sign.*

Let the equation be,

$$f(x) = C_n x^n + C_{n-1} x^{n-1} + \ldots . C_2 x^2 + C_1 x + C_0 = 0.$$

When $x = 0$, the function is negative, by hypothesis, and when $x$ is taken large enough, the function is positive, whether $x$ is positive or negative, since $x^n$ is of even degree. The equation has, therefore, at least two roots, one between zero

and some positive number, the other between zero and some negative number.

Ex. $5x^8 - 13x^5 + 23x^4 + 72x^3 - 19x^2 - 56 = 0$,

must have, at least, two real roots, one positive, and one negative.

**13.** If we could now prove that every equation of an even degree, with its final term *positive*, has a root, we should have established the general proposition that every equation has a root, the basis upon which the whole of the subsequent theory of equations may be established. To know, however, that every equation has not necessarily a *real* root, we do not require to go beyond the lowest of equations of an even degree, that is, quadratics, which may not have any real root when the final term is positive. The question, accordingly, narrows down to this, Have all equations of even degree, with the final term positive, roots, if not real, imaginary, similar to those obtained for equations of the second degree? The investigation of this problem will be given in Chapter II. Before proceeding to that part of the subject, we shall prove a few propositions which are independent of the results of that chapter.

**14.** PROP. VII.—*If a function $f(x)$, and the successive quotients arising from the division, be divided by $x - a$, the remainders, in succession, will be $f(a)$, $f_1(a)$, $\frac{1}{\lfloor 2}f_2(a)$, . . . . $\frac{1}{\lfloor n}f_n(a)$.*

Let the function, which we shall suppose of the $n^{th}$ degree, be divided by $x - a$. As the divisor contains only the first power of $x$, we shall evidently arrive at a remainder independent of $x$, and the quotient, like the dividend, can contain only positive integral powers of $x$. Denoting the quotient by $Q$, and the remainder by $R$, we have

$$f(x) = (x - a)\,Q + R. \qquad [1].$$

If $Q$, which is a function one degree lower than $f(x)$, be in

its turn divided by $x - a$, then, denoting the quotient by $Q_1$, and the remainder by $R_1$, we have

$$Q = Q_1(x - a) + R_1. \qquad [2].$$

Substituting this value of $Q$ in [1], we obtain

$$f(x) = Q_1(x-a)^2 + R_1(x-a) + R.$$

By continuing in this manner to divide each successive quotient by $x - a$, and substituting its equivalent for the symbol for that quotient in the preceding identity, we evidently, after $n$ divisions, shall obtain as quotient the simple factor $C_n$, the coefficient of the first term of $f(x)$, and have

$$f(x) = C_n(x-a)^n + R_{n-1}(x-a)^{n-1} + \,.\,.\, R_2(x-a)^2 + R_1(x-a) + R.$$

In this result, putting $y = x - a$, we have

$$f(a+y) = C_n y^n + R_{n-1}y^{n-1} + \ldots R_2 y^2 + R_1 y + R. \quad [3].$$

Now, by Art. 7, we have, writing the terms in reverse order,

$$f(a+y) = C_n y^n + \frac{1}{\underline{|n-1}} f_{n-1}(a)y^{n-1} + \ldots \frac{1}{\underline{|2}} f_2(a)y^2 + f_1(a)y + f(a) \quad [4].$$

As the coefficients of equal powers of $y$ in these identities must be equal, we find

$$R = f(a),\ R_1 = f_1(a),\ R_2 = \frac{1}{\underline{|2}} f_2(a), \ldots R_{n-1} = \frac{1}{\underline{|n-1}} f_{n-1}(a);$$

that is, *the successive remainders arising from the division by $x - a$ of $f(x)$, and the successive quotients, will be, first, $f(a)$, the value of the function when $a$ is put for $x$; second, $f_1(a)$, the value of the first derived function of $f(x)$, when $a$ is put for $x$, and so on.*

**15.** Cor. 1.—*A function, $f(x)$, is exactly divisible by $x - a$, if $a$ is a root of $f(x) = 0$; for then $f(a) = 0$.*

**16.** Cor. 2.—*Conversely, if $f(x)$ is exactly divisible by $x - a$, then $a$ is a root of the equation $f(x) = 0$; i. e., $f(a) = 0$.*

**17.** In the next proposition will be given a convenient process for finding the successive quotients and remainders without actual division. Yet as this proposition is of considerable practical importance, being the basis of Horner's Method, the student would do well to work one or two examples by dividing in the usual way, and compare the successive remainders with the results obtained by actual substitution in the primitive and derived functions.

Thus suppose we take the function,

$$f(x) = 3x^5 - 13x^4 - 21x^3 + 10x^2 + 91x + 425,$$

then,
$$f_1(x) = 15x^4 - 52x^3 - 63x^2 + 20x + 91,$$

$$\frac{1}{\underline{|2}} f_2(x) = 30x^3 - 78x^2 - 63x + 10,$$

$$\frac{1}{\underline{|3}} f_3(x) = 30x^2 - 52x - 21,$$

$$\frac{1}{\underline{|4}} f_4(x) = 15x - 13,$$

$$\frac{1}{\underline{|5}} f_5(x) = 3.$$

If now we divide $f(x)$ by $x - 3$, we obtain as remainder $-103$, which is the value of $f(3)$, *i. e.*, of the function when 3 is put for $x$. If we divide the first quotient by $x - 3$, we obtain as remainder $-605$, the value of $f_1(3)$, and so on with the other remainders.

## EXERCISES.

**18.** Prove by division that

3 is a root of $x^3 - 7x^2 - 32x + 132 = 0$.
5 " " of $4x^5 - 11x^3 + 20x - 11225 = 0$.
6 " " of $3x^8 - 25x^3 - 1358x^2 - 1625328 = 0$.

**19.** PROP. VIII.—*To find the quotient and remainder when* $f(x) = C_n x^n + C_{n-1} x^{n-1} + \dots . C_2 x^2 + C_1 x + C_0$, *is divided by* $x - a$.

Supposing the division to be performed till we arrive at a remainder independent of $x$, and denoting the quotient by $Q$ and the remainder by $R$, we have

$$f(x) \text{ or } C_n x^n + C_{n-1} x^{n-1} + \ldots C_2 x^2 + C_1 x + C_0 \\ = (x-a)Q + R. \qquad [1].$$

By Art. 14, we know that $R = f(a)$, that is,

$$C_n a^n + C_{n-1} a^{n-1} + \ldots . C_2 a^2 + C_1 a + C_0 = R. \qquad [2].$$

Subtracting [2] from [1], we have

$$C_n(x^n - a^n) + C_{n-1}(x^{n-1} - a^{n-1}) + \ldots . C_2(x^2 - a^2) + C_1(x-a) \\ = (x-a)Q.$$

The binomials $x^n - a^n$, $x^{n-1} - a^{n-1}$, &c., are all divisible by $x - a$, as proved in elementary algebra. Effecting the division, we obtain,

$$\begin{aligned} & C_n(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots . a^{n-2}x + a^{n-1}) \\ + & C_{n-1}(x^{n-2} + ax^{n-3} + \ldots . a^{n-2}) \\ + & \ldots \ldots \ldots \ldots \\ + & C_2(x+a) \\ + & C_1 \;=\; Q. \end{aligned}$$

Arranging this result according to powers of $x$, we have

$$\begin{aligned} C_n x^{n-1} & + (C_n a + C_{n-1}) x^{n-2} + (C_n a^2 + C_{n-1} a + C_{n-2}) x^{n-3} \\ & + (C_n a^3 + C_{n-1} a^2 + C_{n-2} a + C_{n-3}) x^{n-4} + \ldots . \\ & + (C_n a^{n-1} + C_{n-1} a^{n-2} + \ldots . C_3 a^2 + C_2 a + C_1) = Q. \end{aligned}$$

Thus we find that, *in the quotient of $f(x)$ by $x - a$, the coefficient of the first term is $C_n$, and the remaining coefficients are formed in succession by multiplying each term in the preceding coefficient by $a$, and then adding in the corresponding coefficient of $f(x)$.*

**20.** Upon examining the final coefficient of $Q$, namely,

$$C_n a^{n-1} + C_{n-1} a^{n-2} + \ldots . C_3 a^2 + C_2 a + C_1,$$

we see that by applying the process to this coefficient also, and adding in $C_0$, we obtain

$$C_n a^n + C_{n-1} a^{n-1} + \ldots . C_3 a^3 + C_2 a^2 + C_1 a + C_0,$$

which, as before shown, is the remainder, $f(a)$.

**21.** By this process the operation of division, so tedious by the usual method, is reduced to an operation both simple and compact. In practice we proceed as follows: We arrange the coefficients of the function to be divided in a horizontal row, supplying by zeros the coefficients of any terms that may be absent. Supposing $x - a$ to be the divisor, we multiply the first coefficient of the function by $a$, and add in the second; we multiply this result again by $a$, and add in the third coefficient, and so on till we have added in the last coefficient. The final result will be the remainder, which, as we have seen, is also the value of the function when $x = a$. The preceding results will be the coefficients of the quotient, in order, with their proper signs. The first coefficient of the function, which is also that of the quotient, we supply from the left hand of the upper row of coefficients.

Ex. 1.— Required the quotient and remainder arising from the division of

$$x^7 - 5x^6 + 19x^5 - 32x^4 - 63x^3 + 98x^2 + 73x + 42$$

by the binomial $x - 3$.

We proceed as follows:

$$\begin{array}{rrrrrrrrl}
1 & -\ 5 & +\ 19 & -\ 32 & -\ 63 & +\ \ 98 & +\ 73 & +\ 42 & \underline{|\ 3} \\
 & 3 & -\ \ 6 & +\ 39 & +\ 21 & -\ 126 & -\ 84 & -\ 33 & \\
\hline
1 & -\ 2 & +\ 13 & +\ \ 7 & -\ 42 & -\ \ 28 & -\ 11 & +\ \ 9. &
\end{array}$$

Here the quotient is

$$x^6 - 2x^5 + 13x^4 + 7x^3 - 42x^2 - 28x - 11,$$

and the remainder is 9, which is also the value of the given function when $x = 3$.

Ex. 2.— Required the quotient and remainder arising from the division of the function

$$5x^8 - 466x^5 - 637x^4 - 760x^3 - 142x^2 - 370,$$

by the binomial $x - 5$.

$$\begin{array}{rrrrrrrrrl}
5 & +\ 0 & +\ \ \ 0 & -\ 466 & -\ 637 & -\ 760 & -\ 142 & +\ \ 0 & -\ 370 & \underline{|\ 5} \\
 & 25 & +\ 125 & +\ 625 & +\ 795 & +\ 790 & +\ 150 & +\ 40 & +\ 200 & \\
\hline
5 & +\ 25 & +\ 125 & +\ 159 & +\ 158 & +\ \ 30 & +\ \ \ 8 & +\ 40 & -\ 170 &
\end{array}$$

Here the quotient is

$$5x^7 + 25x^6 + 125x^5 + 159x^4 + 158x^3 + 30x^2 + 8x + 40,$$

and the remainder is $-170$.

**22.** By applying the same process to the first quotient, we obtain a second quotient, and a second remainder, which, as shown in Art. 14, is the value of $f_1(x)$ for $x = a$; from the second quotient, in like manner, we obtain a third quotient and a third remainder, which is the value of $\frac{1}{2}f_2(x)$ for $x = a$, and so on. This will be more fully illustrated in Art. 69. As the process thus has its most useful application in finding the value of a given function for an assigned value of $x$, we shall give a few

### EXERCISES.

Evaluate the following functions:

1. $x^5 - 7x^4 + 43x^3 - 78x^2 + 103x - 260$, for $x = 3$.
2. $8x^6 + 13x^5 - 99x^4 - 247x^3 - 387x^2 + 107x + 638$, for $x = 4$.
3. $5x^7 - 646x^4 + 496x^2 - 160x - 121$, for $x = 5$.
4. $x^8 + 73x^3 + 505x^2 + 1059x + 1875$, for $x = -2$.
5. $x^8 - 73x^6 + 54x^4 - 93x^2 + 101$, for $x = 11$.

# CHAPTER II.

## IMAGINARY EXPRESSIONS — CAUCHY'S THEOREM.

**23.** As before adverted to (13), we need not extend our inquiries beyond equations of the second degree to find that it cannot be said, generally, that every equation has a *real* root. We find, in fact, that when certain relations exist among the coefficients of a quadratic, we obtain as roots, not real quantities, but expressions of the form $a + b\sqrt{-1}$, in which $b$ is not zero. Though these, when substituted for $x$ in the given equation, cause it to vanish, and are thus roots by definition (2), yet as they indicate operations to which no arithmetical meaning can be attached, they have received the name of imaginary, or, perhaps more appropriately, impossible roots.

We propose in the present chapter to show that every equation has a root of the form $a + b\sqrt{-1}$, in which $a$ and $b$ are real quantities, positive, negative, or zero. When $b$ is zero, the root is real, in all other cases imaginary.

**24.** Since these imaginary expressions are of constant and unavoidable occurrence in the theory of equations, it may be found convenient to give a summary of the chief results arising from the conventions adopted in regard to them.

The first convention is that we regard $\pm\sqrt{-a^2}$ as equivalent to $\pm a\sqrt{-1}$, since thus we introduce but one imaginary symbol into our investigations, namely $\sqrt{-1}$. The whole expression $a + b\sqrt{-1}$, consisting of a real part $a$, and another real part $b$ affected by the sign of an impossible operation, we consider as imaginary on account of the presence of this latter part. We also regard such terms as $b\sqrt{-1}$ as subject to the ordinary rules that govern algebraical transformations.

**25.** From these conventions we obtain the following results:

I. The sum or difference of expressions of the form $A + B\sqrt{-1}$, have the same form; thus,

$$(a \pm b\sqrt{-1}) \pm (a' + b'\sqrt{-1}) =$$
$$(a \pm a') \qquad \pm (b \pm b')\sqrt{-1},$$

which is of the form $A + B\sqrt{-1}$.

II. The product of expressions of the form $A + B\sqrt{-1}$ is of the same form; thus,

$$(a \pm b\sqrt{-1}) \times (a' \pm b'\sqrt{-1}) = (aa' - bb') \pm (a'b + ab')\sqrt{-1}.$$

III. The quotient of one expression of the form $A + B\sqrt{-1}$, by another of that form, is still of the same form; thus,

$$\frac{a \pm b\sqrt{-1}}{a' \pm b'\sqrt{-1}} = \frac{(a \pm b\sqrt{-1})(a' \mp b'\sqrt{-1})}{(a' \pm b'\sqrt{-1})(a' \mp b'\sqrt{-1})} =$$
$$\frac{(aa' \pm bb') \pm (a'b - ab')\sqrt{-1}}{a'^2 + b'^2}$$

which is still of the given form.

**26.** From the result obtained in II, we infer that any positive integral power of $a + b\sqrt{-1}$ is of the same form; and may thence infer that if $a + b\sqrt{-1}$ be substituted for $x$ in a rational integral function of $x$, whether the coefficients be all real, or any of them imaginary, we shall obtain an expression of the form $P + Q\sqrt{-1}$.

**27.** Two imaginary expressions $a + b\sqrt{-1}$, and $a - b\sqrt{-1}$, which differ only in the sign of the imaginary part, are said to be *conjugate.*

Hence the sum and the product of two conjugate imaginary expressions are real. This we see exemplified in the imaginary roots, $a + b\sqrt{-1}$, and $a - b\sqrt{-1}$, of a quadratic, their sum $2a$, with changed sign, being the coefficient of $x$, and their product, $a^2 + b^2$, the final term.

**28.** The positive value of the square root of this product, that is, $\sqrt{a^2 + b^2}$, is called the *modulus* of each of the expressions $a + b\sqrt{-1}$, and $a - b\sqrt{-1}$. The modulus of a real quantity is, by the above definition, the positive value of the quantity itself.

When, for any purpose, we desire to compare imaginary expressions with each other or with real quantities in regard to magnitude, this can be done only by comparing the moduli of the expressions.

**29.** In order that an imaginary expression be zero, it is necessary and sufficient that its modulus be zero: for in order that $\sqrt{a^2 + b^2}$ may be zero, we must have $a = 0$, and $b = 0$.

**30.** The product of two quantities has for modulus the product of their moduli; thus,

$$(a + b\sqrt{-1})\,(a' + b'\sqrt{-1}) = \\ (aa' - bb') + (ab' + a'b)\sqrt{-1},$$

and the modulus of this is,

$$\sqrt{\{(aa' - bb')^2 + (ab' + a'b)^2\}} = \sqrt{(a^2 + b^2)\,(a'^2 + b'^2)},$$

which is the product of the moduli of the original expressions.

In a similar manner it may be shown that the modulus of the quotient of two quantities, is the quotient of the modulus of the dividend by the modulus of the divisor.

**31.** If we raise $\sqrt{-1}$ to the second power, we obtain $-1$; this multiplied by $\sqrt{-1}$, gives the third power $-\sqrt{-1}$; This again multiplied by $\sqrt{-1}$, gives the fourth power $+1$. Upon proceeding to higher powers, we obtain a recurrence of the preceding results, $\sqrt{-1}$, $-1$, $-\sqrt{-1}$, $+1$. Every whole number must be of one of the four forms, $4r$, $4r + 1$, $4r + 2$, $4r + 3$, since, when divided by 4, it must leave as remainder, 0, 1, 2, or 3. We have, therefore, all possible integral powers of $\sqrt{-1}$ in the four forms,

$$(\sqrt{-1})^{4r} = 1, \quad (\sqrt{-1})^{4r+1} = \sqrt{-1}, \quad (\sqrt{-1})^{4+2} = -1, \\ (\sqrt{-1})^{4r+3} = -\sqrt{-1}.$$

**32.** In elementary algebra is given the formula for the square of an expression of the form $a + b\sqrt{-1}$,

$$\sqrt{a+b\sqrt{-1}} = \left\{\tfrac{1}{2}(\sqrt{a^2+b^2}+a)\right\}^{\frac{1}{2}} + \left\{\tfrac{1}{2}(\sqrt{a^2+b^2}-a)\right\}^{\frac{1}{2}}\sqrt{-1}.$$

From this we may obtain the square roots of $\pm\sqrt{-1}$, by putting $a = 0$, $b = \pm 1$. Thus we obtain

$$\sqrt{+\sqrt{-1}} = \pm\frac{1+\sqrt{-1}}{\sqrt{2}}, \quad \sqrt{-\sqrt{-1}} = \pm\frac{1-\sqrt{-1}}{\sqrt{2}}.$$

**33.** Before proceeding to the general proposition that every equation has a root, it will be found convenient to discuss certain equations of very simple form.

PROP. I.—*Each of the equations,*

$$x^n = \pm 1, \qquad x^n = \pm\sqrt{-1},$$

*has a root, real or imaginary.*

I. $x^n = +1$. This equation evidently has a real root, whether $n$ be an odd or an even number, since $x = 1$ satisfies it in either case.

II. $x^n = -1$. When $n$ is an odd number, this equation is obviously satisfied by $x = -1$.

When $n$ is an even number, it must be of the form $2m$; that is, we may put the equation under the form $y^{2m} = -1$. Taking the square root of both members, we obtain $y^m = \pm\sqrt{-1}$, an equation of the forms we are about to consider.

III. $x^n = +\sqrt{-1}$. When $n$ is an odd number, it must be of the form $4r+1$, or $4r+3$. In the first case, $x = +\sqrt{-1}$ is a root, since (31), $(+\sqrt{-1})^{4r+1} = +\sqrt{-1}$; in the second case, $x = -\sqrt{-1}$ is a root, since $(-\sqrt{-1})^{4r+3} = -\sqrt{-1}$.

When $n$ is an even number, it must be either some power of 2, or some power of 2 multiplied by an odd number. In the first case suppose $n = 2^m$, that is, put the equation under the form $y^{2^m} = +\sqrt{-1}$. We can now obtain a

value of $y$ by $m$ successive extractions of the square root, which (32) will yield a result of the form $a + b\sqrt{-1}$. In the second case suppose $n = mp$, where $m$ is an odd number, and $p$ some power of 2. By putting $y = x^p$, the equation may be put under the form $y^m = +\sqrt{-1}$, a root of which, as shown above, must be $y = \pm\sqrt{-1}$, or $x^p = \pm\sqrt{-1}$, the upper sign being taken when $m$ is of the form $4r+1$, the lower when it is of the form $4r+3$. We can now find, as above, a root of $x^p = \pm\sqrt{-1}$, $p$ being a power of 2.

IV. $x^n = -\sqrt{-1}$. Proceeding as in III, we find, when $n$ is odd, $x = -\sqrt{-1}$, or $+\sqrt{-1}$, according as $n$ is of the form $4r+1$, or $4r+3$. When $n$ is an even number, we may put it equal to $mp$, $m$ being an odd number, and $p$ some power of 2, and so find a root of the form $a + b\sqrt{-1}$.

Thus, in every case, an equation of any of the given forms has a root coming under the general form $a + b\sqrt{-1}$.

**34.** Prop. II.—*Every rational integral equation has a root of the general form* $a + b\sqrt{-1}$.

Let $f(x) = C_n x^n + C_{n-1}x^{n-1} + \dots C_2x^2 + C_1x + C_0$,

in which the coefficients $C_n$, $C_{n-1}, \dots C_1$, $C_0$, may be either real or imaginary.

If, in this function, we substitute $a + b\sqrt{-1}$ for $x$, $a$ and $b$ being real, we shall obtain (26) a result of the form $P + Q\sqrt{-1}$, $P$ and $Q$ being real. Now, in order that $a + b\sqrt{-1}$ may be a root of $f(x) = 0$, this result must be zero, that is, $P$ and $Q$ must be simultaneously zero, and therefore the modulus $\sqrt{P^2 + Q^2}$ also zero. It is required to show that some value of $a + b\sqrt{-1}$ must exist, for which $\sqrt{P^2+Q^2}$ becomes zero. For, if it could not become zero, there would be some value below which it could not be diminished. But it will be proved that whatever value of $\sqrt{P^2+Q^2}$, different from zero, can be obtained, a value still smaller can be obtained by making a suitable change in the expression that is substituted for $x$ in the function. $\sqrt{P^2+Q^2}$, therefore, must be capable of becoming zero for some value of

$a + b\sqrt{-1}$, that is, the function must become zero for some value of $x$.

Let us suppose that for some assigned value of $x$, as $x = a + b\sqrt{-1}$, we obtain

$$f(x) = P + Q\sqrt{-1}. \qquad [1]$$

in which $P$ and $Q$ are *not* both zero.

If we change $a + b\sqrt{-1}$ to $a + b\sqrt{-1} + h$, that is, $x$ to $x + h$, we have, Art. 7,

$$f(x+h) = f(x) + f_1(x)h + f_2(x)h^2\frac{1}{\lfloor 2} + \ldots f_n(x)h^n\frac{1}{\lfloor n}. \quad [2].$$

In [1] we have, for $x = a + b\sqrt{-1}$, $f(x) = P + Q\sqrt{-1}$; for this value of $x$ some of the derived functions $f_1(x)$, $f_2(x)$, &c., in [2] may possibly become zero, but they cannot all become zero, $f_n(x)$, at least, which (7) does not contain $x$, must remain. Suppose $h^m$ to be the lowest power of $h$ whose coefficient does not vanish in [2]. Having denoted the value of $f(x)$ by $P + Q\sqrt{-1}$, so we may put $P' + Q'\sqrt{-1}$ for the value of $f(x+h)$, and $R + S\sqrt{-1}$, in which $R$ and $S$ are not both zero, for the value of that derived function which is the coefficient of $h^m$.

Substituting these values in [2], we have,

$$P' + Q'\sqrt{-1} = (P + Q\sqrt{-1}) + (R + S\sqrt{-1})h^m$$
$$+ \text{ terms in } h^{m+1},\ h^{m+2}, \text{ \&c.} \quad [3].$$

As we may assign any value we please to $h$, we may replace it by $kt$, where $k$ is a real positive quantity, and $t$ is such that (33) $t^m$ may be $+1$, or $-1$, as we choose. Hence we can make $P' + Q'\sqrt{-1} = (P + Q\sqrt{-1} \pm (R + S\sqrt{-1})k^m$ + terms in higher powers of $k$.

In such an equation the sum of the real terms in one member must be equal to the sum of the real terms in the other, and the imaginary terms in the one to the imaginary terms in the other:

$$\therefore \quad P' = P \pm Rk^m + \text{ terms in } k^{m+1},\ k^{m+2}, \text{ \&c.}$$
$$Q' = Q \pm Sk^m + \text{ terms in } k^{m+1},\ k^{m+2}, \text{ \&c.}$$
$$\therefore \quad P'^2 + Q'^2 = (P^2 + Q^2) \pm 2(PR + QS)k^m + \text{ other}$$
$$\text{terms in } k^{m+1}, \text{ \&c.} \quad [4].$$

Now $k$ may be taken so small that (Art. 6, Cor. 1) the sum of all the terms after $P^2 + Q^2$ will have the same sign as $2(PR + QS)k^m$, provided $PR + QS$ be not zero.

First, supposing $PR + QS$ is not zero, then the sign of $(P'^2 + Q'^2) - (P^2 + Q^2)$ is the same as that of $\pm 2(PR + QS)k^m$, when $k$ is taken small enough, and this sign we can always make negative by taking $t$ such that $t^m = -1$, when $PR + QS$ is positive, and *vice versa*. We can thus always make $P'^2 + Q'^2$ less than $P^2 + Q^2$.

If $PR + QS$ be zero, then we choose $t$ so as to make negative the first term after $2(PR + QS)k^m$ that does not vanish in [4], and as before we obtain $(P'^2 + Q'^2) - (P^2 + Q^2) =$ a negative quantity, that is, we have $\sqrt{P'^2 + Q'^2} < \sqrt{P^2 + Q^2}$.

Thus, whatever be the value of the modulus $\sqrt{P^2 + Q^2}$, a modulus $\sqrt{P'^2 + Q'^2}$ of smaller value may be obtained by making a suitable change in the expression that is substituted for $x$ in $f(x)$; there must, therefore, be some expression which, substituted for $x$, will make the modulus zero.

**35.** Prop. III. — *The values of $a$ and $b$ in the expression $a + b\sqrt{-1}$, which, when put for $x$ in $f(x)$, causes it to vanish, are finite.*

We may write $f(x)$ as follows:

$$f(x) = C_n x^n \left\{1 + \frac{C_{n-1}}{C_n x} + \frac{C_{n-2}}{C_n x^2} + \dots \frac{C_1}{C_n x^{n-1}} + \frac{C_0}{C_n x^n}\right\}.$$

Substituting $a + b\sqrt{-1}$ for $x$ in the second member, it becomes,

$$C_n(a + b\sqrt{-1})^n \left\{1 + \frac{C_{n-1}}{C_n(a+b\sqrt{-1})} + \frac{C_{n-2}}{C_n(a+b\sqrt{-1})^2} + \dots \frac{C_0}{C_n(a+b\sqrt{-1})^n}\right\}.$$

Selecting any term from the series within the brackets, the third, for example, we have,

$$\frac{C_{n-2}}{C_n(a+b\sqrt{-1})^2} = \frac{C_{n-2}(a-b\sqrt{-1})^2}{C_n(a^2+b^2)^2}$$
$$= \frac{C_{n-2}(a^2-b^2)}{C_n(a^2+b^2)^2} - \frac{2C_{n-2}ab\sqrt{-1}}{C_n(a^2+b^2)^2} = A + B\sqrt{-1}, \text{ say.}$$

In this it is obvious that $A$ and $B$ diminish without limit, as $a$ and $b$ increase without limit. Denoting the value of $f(x)$, for $x = a + b\sqrt{-1}$, by $P + Q\sqrt{-1}$; we have

$$P + Q\sqrt{-1} = C_n(a+b\sqrt{-1})^n\{1+A'+B'\sqrt{-1}\},$$

in which $A'$ and $B'$ diminish without limit, as $a$ and $b$ increase without limit. In the same way we have, for $x = a - b\sqrt{-1}$,

$$P - Q\sqrt{-1} = C_n(a-b\sqrt{-1})^n\{1 + A' - B'\sqrt{-1}\};$$
$$\therefore\ P^2 + Q^2 = C_n^2(a^2+b^2)^n\{(1+A')^2 + B'^2\}.$$

This result increases without limit, when $a$ and $b$ increase without limit; for then the factor $(a^2+b^2)^n$ increases without limit, and the factor $\{(1+A')^2 + B'^2\}$ tends towards unity, as $A'$ and $B'$ decrease without limit. Thus the modulus $\sqrt{P^2+Q^2}$, cannot become zero when $a$ or $b$, or both of them, are indefinitely great.

By the present proposition, therefore, in conjunction with that of Art. 34, we find that every rational integral equation, whether its coefficients be real or imaginary, has a root coming under the general form $a + b\sqrt{-1}$, where $a$ and $b$ are finite quantities, either, or both, positive, negative, or zero.

An irrational equation, as, for example,

$$\sqrt{x-4} - \sqrt{x-1} - 1 = 0,$$

may be incapable of being satisfied by any quantity, real or imaginary, if we regard the signs of the roots indicated as being controlled by the plus and minus signs prefixed. The above equation, when rationalized and solved in the usual manner, gives $x = 5$, a value that does not satisfy the proposed equation, but does satisfy the equation

$$-\sqrt{x-4} + \sqrt{x-1} - 1 = 0.$$

# CHAPTER III.

## GENERAL PROPERTIES OF EQUATIONS, DEPENDENT ON THE PRINCIPLE THAT EVERY EQUATION HAS A ROOT.

**36.** PROP. I.—*Every equation has as many roots as it has dimensions, and no more.*

Let $f(x) = C_n x^n + C_{n-1} x^{n-1} + \dots C_2 x^2 + C_1 x + C_0 = 0,$ [1]

be an equation of the $n^{th}$ degree, it has $n$ roots, and no more. By Art. 34, $f(x) = 0$ has a root $a_1$, real or imaginary, and is therefore (15) divisible by $x - a_1$. Hence we may put,

$$f(x) = (x - a_1)\,\phi_1(x), \qquad [2]$$

where $\phi_1(x)$ is a rational integral function of the $(n-1)^{th}$ degree. If $\phi_1(x)$, again, be equated to zero, it must also have some root $a_2$, and is, therefore, divisible by $x - a_2$, so that we have $\phi_1(x) = x - x_2)\,\phi_2(x)$, where $\phi_2(x)$ is a function of the $(n-2)^{th}$ degree. Substituting this value of $\phi_1(x)$ in [2], we have,

$$f(x) = (x - a_1)\,(x - a_2)\,\phi_2(x). \qquad [3].$$

In like manner, $\phi_2(x)$, equated to zero, has some root $a_3$, and has, therefore, a binomial divisor $(x - a_3)$, divided by which it will yield as quotient a function of the $(n-3)^{rd}$ degree, which we may denote by $\phi_3(x)$; we have, therefore,

$$f(x) = (x-a_1)\,(x-a_2)\,(x-a_3)\,\phi_3(x). \qquad [4].$$

It is evident that a continuance of the process will lead at last to the simple factor $C_n$, and that there must be $n$ binomial factors $(x-a_1)$, $(x-a_2)$, . . . $(x-a_n)$; therefore

$$f(x) = C_n\,(x-a_1)\,(x-a_2)\,(x-a_3)\;.\;.\;.\;(x-a_n) = 0.$$

Hence the equation $f(x) = 0$ has $n$ roots, since the function vanishes when any one of the quantities $a_1, a_2, a_3, \dots a_n$, is put for $x$.

Nor can the equation have more than $n$ roots; for when we put for $x$ a quantity $b$, which is not one of the quantities $a_1, a_2, a_3, \ldots a_n$, we have

$$f(b) = C_n(b-a_1)(b-a_2)(b-a_3)\ldots(b-a_n),$$

the second member of which cannot be zero, since none of the factors is zero.

**37.** Observe that it is not asserted that the $n$ roots $a_1, a_2, a_3, \ldots a_n$, of an equation of the $n^{th}$ degree are necessarily different; some, or all of them, may be equal. What the reasoning shows is that the first member of an equation of the $n^{th}$ degree may be resolved into the product of $n$ binomial factors, each of which, equated to zero, will furnish a root. It is found convenient to say that the equation has as many roots as dimensions, without regard to the relative magnitudes of the roots.

**38.** COR. 1.—From this proposition it is evident that, when a root $a_1$ of the equation $f(x) = 0$ is known, we can obtain an equation containing all the remaining roots, by dividing $f(x)$ by $x - a_1$, and equating the quotient to zero, which latter equation is of one degree lower than $f(x) = 0$. When two roots, $a_1$ and $a_2$, are known, we can, by dividing $f(x)$ successively by $(x-a_1)$ and $(x-a_2)$, or at once by their product, obtain a function of $x$, two degrees lower than $f(x)$, which quotient, equated to zero, will contain all the remaining roots; and similarly for any number of roots.

Ex. $x^3 - 18x^2 + 101x - 180 = 0$ has a root $x = 4$.

Dividing by $x - 4$, we obtain $x^2 - 14x + 45$, which equated to zero gives two other roots, 5, and 9.

**39.** COR. 2.—If a rational integral function of $x$ of the $n^{th}$ degree can become zero for more than $n$ values of $x$, then every coefficient in the function must be equal to zero, and the function must be zero for any value of $x$.

For, otherwise, the function equated to zero would have more than $n$ roots; which, by the present proposition, is

impossible, unless the coefficients be each equal to zero; and if these coefficients be each zero, the function is evidently zero for every value of $x$.

**40.** COR. 3.—If $f(x)$ be of the $n^{th}$ degree, it has $\frac{n(n-1)\ldots(n-r+1)}{\lfloor r}$ factors of the $r^{th}$ degree.

For it has been shown that $f(x)$ has $n$ factors of the first degree. The product of every two of these simple factors is a factor of the second degree; so there must be as many factors of the second degree as there are possible combinations of $n$ things taken two at a time, that is, $\frac{n(n-1)}{\lfloor 2}$. In like manner, with the $n$ simple factors may be formed as many factors of $r^{th}$ degree as there are possible combinations of $n$ things taken $r$ at a time, that is, $f(x)$ admits of $\frac{n(n-1)\ldots\ldots(n-r+1)}{\lfloor r}$ factors of the $r^{th}$ degree. These factors, of any degree, will not, of course, be all different, unless the simple factors are all different. Thus the expression,

$$x^3 - 18x^2 + 101x - 180 = (x-9)(x-5)(x-4)$$

may be regarded as the product of three binomial factors, or as the product of any one of these factors and a function of the second degree produced by the remaining two factors.

**41.** By this proposition we find that *every rational integral function of $x$ may be regarded as the product of as many binomial factors as it has dimensions.* But to resolve a given function into its component factors, we require to know the roots of that function equated to zero, *i. e.*, the resolution of a function of $n$ dimensions into factors, requires the solution of an equation of the $n^{th}$ degree. The converse operation, however, of composing a function from given simple elements, or constructing an equation that shall have for its roots given quantities, is very easily accomplished. The roots $a_1$, $a_2$, $a_3$, &c., being given, we form the binomial factors $(x-a_1)$, $(x-a_2)$, $(x-a_3)$, &c., the product of which, equated to zero, is the

equation required. It is required, for example, to form the equation the roots of which shall be, 7, 5, $-3$, $3+\sqrt{-2}$, $3-\sqrt{-2}$.

The product of the binomial factors found from these is,

$$(x-7)(x-5)(x+3)(x-3+\sqrt{-2})(x-3-\sqrt{-2})$$

Multiplying out, and equating the result to zero, we have,

$$x^5 - 15x^4 + 64x^3 + 12x^2 - 641x + 1155 = 0,$$

which is the required equation.

EXERCISES.

1. The roots of $x^4 - x^3 - 35x^2 + 129x - 126 = 0$, are 3, 3, 2, $-7$; express the first member as the product of binomial factors.

2. The roots of $x^5 - 6x^4 - 37x^3 + 106x^2 + 456x + 320 = 0$, are 8, 5, $-4$, $-2$, $-1$; express the first member as the product of binomial factors.

3. Form the equation whose roots are 7, 4, $-11$.

4. Form the equation whose roots are 5, 4, 2, 1.

5. Form the equation whose roots are 7, 3, $-4$, $-10$.

6. Form the equation whose roots are 10, 8, 5, and $1 \pm \sqrt{-3}$.

**42.** PROP. II.—*An equation with real coefficients has its imaginary roots, if any, in conjugate pairs.*

Let $f(x) = C_n x^n + C_{n-1}x^{n-1} + \dots C_2x^2 + C_1x + C_0 = 0$,

where $C_n$, $C_{n-1}, \dots C_0$, are all real, have a root $a + b\sqrt{-1}$, then is $a - b\sqrt{-1}$ also a root.

Let $a + b\sqrt{-1}$ be substituted for $x$ in the equation, then (26) we obtain a result of the form $P + Qb\sqrt{-1}$, where $P$ and $Q$ contain only even powers of $b\sqrt{-1}$. For, when an expression $(a + b\sqrt{-1})^n$ is expanded by the Binomial Theorem, the even powers of $b\sqrt{-1}$ give rise to real quantities, so that only odd powers of $b$ are affected by the symbol $\sqrt{-1}$; and, as the coefficients of $f(x)$ are supposed to be real, the symbol $\sqrt{-1}$ cannot occur in $f(a + b\sqrt{-1}$ ex-

cept with odd powers of $b$. Therefore, if $P + Qb\sqrt{-1}$ denote the value of $f(x)$ for $x + a + b\sqrt{-1}$, we can obtain the value of the same function for $x = a - b\sqrt{-1}$ by simply changing the sign of $b$ in the former result. Thus if

$$f(a + b\sqrt{-1}) = P + Qb\sqrt{-1}, \text{ then}$$
$$f(a - b\sqrt{-1}) = P - Qb\sqrt{-1}.$$

But if $a + b\sqrt{-1}$ be a root of the equation $f(x) = 0$, we have

$$P + Qb\sqrt{-1} = 0,$$

and as no part of $P$ can be cancelled by $Qb\sqrt{-1}$, this result requires that $P = 0$, and also $Q = 0$. Hence, also,

$$P - Qb\sqrt{-1} = 0,$$

that is, $a - b\sqrt{-1}$ is also a root of $f(x) = 0$.

**43.** COR. 1.—*Every function of an even degree may be regarded as the product of real quadratic factors,* whatever be the character of the roots. For if the function, equated to zero, have a root $a + b\sqrt{-1}$, $b$ not being zero, it must also have its conjugate $a - b\sqrt{-1}$ as a root. The function must therefore be divisible by $(x - \overline{a + b\sqrt{-1}})(x - \overline{a - b\sqrt{-1}})$ $= x^2 - 2ax + a^2 + b^2$, a real quadratic factor.

**44.** COR. 2.—By a course of reasoning similar to that employed above, it may be shown that, *if an equation have a root of the form $a + \sqrt{b}$, it must also have a root $a - \sqrt{b}$.*

EXERCISES.

1. One root of $x^4 - 27x^2 + 90x - 36 = 0$ is $3 + \sqrt{-3}$; find the remaining roots.

2. One root of $x^4 + 5x^3 - 25x^2 - 140x - 26 = 0$ is $-5 + \sqrt{-1}$; find the remaining roots.

3. One root of $6x^4 + 37x^3 + 53x^2 - 89x - 30 = 0$ is $\frac{1}{2}(7 - \sqrt{-11})$; find the remaining roots.

4. One root of $5x^4 - 3x^3 + 6x^2 - x + 3 = 0$ is $\frac{1}{2}(1 + \sqrt{-3})$; find the remaining roots.

5. One root of $x^4 - 43x^2 - 24x + 108 = 0$ is $6 \cdot 6457513 = 4 + \sqrt{7}$; find the remaining roots.

**45.** Prop. III. — *To determine the relations that exist between the coefficients and the roots of an equation.*

Let $f(x) = x^n + C_{n-1}x^{n-1} + \ldots . C_2x^2 + C_1x + C_0 = 0.$

It has been proved that if $a_1, a_2, a_3, \ldots a_n$, be the roots of $f(x) = 0$, then

$$f(x) = (x-a_1)(x-a_2)(x-a_3)\ldots .(x-a_n).$$

By actual multiplication we find

$$(x-a_1)(x-a_2) = x^2 - (a_1+a_2)x + a_1a_2,$$

$$(x-a_1)(x-a_2)(x-a_3) = x^3 - (a_1+a_2+a_3)x^2 + (a_1a_2+a_1a_3+a_2a_3)x - a_1a_2a_3,$$

$$(x-a_1)(x-a_2)(x-a_3)(x-a_4) = x^4 - (a_1+a_2+a_3+a_4)x^3 + (a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4)x^2 - (a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4)x + a_1a_2a_3a_4.$$

In these results we observe the following laws to hold:

I. The number of terms in the product on the right hand is greater by one than the number of binomial factors.

II. The exponent of $x$ diminishes by unity in each term, from the first, where it is the same as the number of binomial factors, to the last, where it is zero.

III. The coefficient of the first term is unity, that of the second term is the sum of the second letters in the binomial factors, that of the third term is the sum of the products of these letters taken two at a time, that of the fourth term is the sum of the products of these letters taken three at a time, and so on; finally, the last term is the product of all these letters.

We now proceed to prove that, if these laws hold good for $(n-1)$ factors, they must hold good for $n$ factors, and thus universally. Let us suppose that

$$(x-a_1)(x-a_2)\ldots(x-a_{n-1}) = x^{n-1} + S_1x^{n-2} + S_2x^{n-3} + \ldots S_{n-1}$$

where $S_1$ denotes the sum of the letters $-a_1, -a_2, -a_3, \ldots -a_{n-1}$,
" $S_2$ " " " products of these letters two by two,
" $S_3$ " " " do. do. three by three,
. . . . . . . . . .
" $S_{n-1}$ " " " product of all the $(n-1)$ letters.

Multiplying both sides of this identity by another factor $x - a_n$, the result will be,

$$(x-a_1)(x-a_2)\ldots(x-a_n) = x^n + S_1 - a_n)x^{n-1} + (S_2 - S_1 a_n)x^{n-2} + \ldots - S_{n-1}a^n.$$

Here it is obvious that the first and second laws still hold. As regards the coefficients, that of the first term is still unity,

that of the 2d term, $S_1 - a_n = -a_1, -a_2, -a_3, \ldots -a_n$
= the sum of all the letters, $-a_1, -a_2, \ldots -a_n$.
" " 3d " $S_2 - S_1 a_n = S_2 + a_n(a_1 + a_2 + a_3 + \ldots + a_n)$
= the sum of the products of all the letters two by two.
" " 4th " $S_3 - S_2 a_n = S_3 - a_n(a_1 a_2 + a_2 a_3 + \ldots\ldots)$
= the sum of the products of all the letters three by three.
" " $n$th " $-S_{n-1}a_n$ = the product of all the letters.

If, then, these laws hold good for $(n-1)$ factors, we see that they hold good for $n$ factors. But we know that they hold good for four factors; they must, therefore, hold for five factors, and so on for any number of factors.

Therefore, $a_1, a_2, a_3, \ldots a_n$ being the roots of

$f(x) = x^n + C_{n-1}x^{n-1} + C_{n-2}x^{n-2} + \ldots C_1 x + C_0 = 0$, we have also

$$f(x) = (x-a_1)(x-a_2)(x-a_3)\ldots\ldots(x-a_n) = 0$$
$$= x^n + S_1 x^{n-1} + S_2 x^{n-2} + \ldots\ldots S_{n-1}x + S_n = 0,$$

and equating coefficients of like powers of $x$, we have

$$C_{n-1} = S_1, \quad C_{n-2} = S_2, \; \ldots\ldots \; C_1 = S_{n-1}, \quad C_0 = S_n,$$

that is, *the coefficient of the second term is equal to the sum of the roots with their signs changed, the coefficient of the third*

*term to the sum of the products of every two of the roots with their signs changed, and so on, the coefficient of the $r^{th}$ term being equal to the sum of the products of every $(r-1)$ of the roots with their signs changed, the last term being the product of them all taken with changed signs.*

We may also enunciate these relations thus:

$-C_{n-1}$ = sum of the roots; $C_{n-2}$ = sum of the products of every two of the roots; $-C_{n-3}$ = sum of the products of every three of the roots; and generally $(-1)^r C_{n-r}$ = sum of the products of every $r$ of the roots.

**46.** It is obvious that if the coefficient of the first term be other than unity, we must take the other coefficients divided by that leading coefficient when we wish to obtain the sum of the roots, &c.

**47.** As these relations between the roots and coefficients furnish $n$ independent equations involving the roots, it might naturally be supposed that by eliminating $n-1$ of the roots we should obtain an equation from which to determine the remaining root. By performing the elimination, we do, in fact, obtain such an equation, but find that we have merely reproduced the original equation with another symbol for the unknown quantity.

Let $a$, $b$, $c$, for example, denote the roots of the cubic

$$x^3 + px^2 + qx + r = 0,$$

then we have

$$-p = a + b + c, \quad q = ab + ac + bc, \quad -r = abc.$$

To eliminate $b$ and $c$ between these equations, we multiply the first by $a^2$, the second by $a$, and add in the third; we thus obtain,

$$a^3 + pa^2 + qa + r = 0.$$

The necessity of this result is readily perceived when we consider, that of the three symbols $a$, $b$, $c$, any one, as $a$, represents any one of the three roots without distinction, and the equation in $a$, which we obtain by eliminating $b$ and $c$, must allow of three values for $a$, that is, must be a cubic. Yet,

though these relations between the roots and coefficients do not enable us to determine the roots, they do enable us to discover many important properties of roots, as will be seen in subsequent chapters.

**48.** The following proposition is independent of the principle that every equation has a root, yet it is most conveniently inserted here, as a lemma to a useful corollary from the preceding proposition.

PROP. IV.—*An equation in which the first coefficient is unity, and the remaining coefficients integers, cannot have a rational fractional root.*

Let $\qquad x^n + C_{n-1}x^{n-1} + \ldots . C_2x^2 + C_1x + C_0 = 0$

be such an equation; and, if possible, let it have as a root the rational fraction $\frac{a}{b}$, $a$ and $b$ being prime to each other. By substituting $\frac{a}{b}$ for $x$, and multiplying throughout $b^{n-1}$, we have,

$$\frac{a^n}{b} + C_{n-1}a^{n-1} + \ldots + C_2a^2b^{n-3} + C_1ab^{n-2} + C_0b^{n-1} = 0.$$

If we transpose the first term $\frac{a^n}{b}$ to the right hand, we have the remaining terms, which must form an integral expression, since $a$ and $b$, and all the coefficients, are integers, equal to $-\frac{a^n}{b}$, which is not an integer. This is impossible, therefore $\frac{a}{b}$ cannot be a root of the proposed equation.

**49.** Such an equation may, however, have incommensurable real roots, or imaginary roots of the form $a + b\sqrt{-1}$, in which $a$ and $b$ are incommensurable. For example, the equation

$$x^3 + 5x^2 - 10x - 8 = 0$$

has a root $x = 2$. Dividing by $x - 2$, we obtain

$$x^2 + 7x + 4 = 0,$$

which has for roots $x = -\frac{1}{2}(7 \pm \sqrt{33})$, which are fractional, but not rational.

**50.** COR.—Since (45) the final term is the product of all the roots, it follows, by the present proposition, that *in any equation having unity for its leading coefficient, and the remaining coefficients integers, all the rational roots must be integral factors of the final term.* Also, if this term, $C_0$, be a prime number, the equation cannot have rational roots, if $\pm 1$, or $\pm C_0$, be not roots.

The equation $x^5 - 13x^2 + 9x - 11 = 0$, for example, must (12) have, at least, one real root; but since, upon trial, we find that neither $\pm 1$, nor $\pm 11$, is a root, we know that it has no rational root.

**51.** Since, as will presently be shown, we can always transform an equation, having a leading coefficient other than unity, into one having unity for leading coefficient, processes, that will readily suggest themselves, have been deduced from the above property, by which we can, in any case, discover whether an equation has rational roots. Such roots, however, occur rarely in equations that actually present themselves for solution; and, when they do occur, will necessarily be discovered in the course of the process of analysis hereafter to be explained. It is, therefore, not necessary to make a special search for rational roots. The following easy examples given by way of illustration, may be solved by the guidance of the property given in Art. 50, and by means of the process illustrated in Art. 21.

### EXERCISES.

Find the rational roots of the following equations:

1. $x^3 - 6x^2 + 10x - 3 = 0.$
2. $x^3 - 17x - 40 = 0.$
3. $x^4 - 4x^3 - 56x^2 + 217x + 28 = 0.$
4. $x^4 - 2x^3 - 18x^2 + x + 70 = 0.$
5. $x^5 - 37x^2 - 921x - 918 = 0.$

**52.** DEFINITIONS.—A *Variation* is a change of sign in passing from one term to another; a *Permanence* is the continuation of the same sign in two consecutive terms.

Thus, in the function,

$$5x^7 - 7x^6 - 13^5 + x^4 - 17x^3 + 3x^2 + 6x + 10,$$

there are four variations and three permanences; the sum of the number of variations, and of the number of permanences being equal to the number expressing the degree of the function, as will evidently be the case in a function of any degree. By changing $x$ to $-x$, the function becomes

$$-5x^7 - 7x^6 + 13x^5 + x^4 + 17x^3 + 3x^2 - 6x + 10,$$

the variations being changed to permanences, and *vice versa.*

**53.** PROP. V.—*An equation, complete or incomplete, cannot have more positive roots than variations of sign, and a complete equation cannot have more negative roots than it has permanences of sign.*

Let the signs of the proposed function be, for example,

$$+ \; - \; - \; + \; - \; + \; + \; + \; - \; - \; + \; ;$$

we shall show that upon multiplying by a factor $x - a$, there will be in the result at least one more change of sign than in the original function. For, having to multiply by a binomial whose first sign is positive, and the second negative, we obtain, giving merely the signs that occur in the process,

$$\begin{array}{cccccccccccc}
+ & - & - & + & - & + & + & + & - & - & + & \\
 & - & + & + & - & + & - & - & - & + & + & - \\
\hline
+ & - & \mp & + & - & + & \pm & \pm & - & \mp & + & -
\end{array}$$

writing the double sign in the result wherever the sign of any term is ambiguous.

Upon comparing the series of signs in the result with the series in the original function, we find

(1). For every group of permanences there is a corresponding group of ambiguities.

(2). The signs before each ambiguity, or group of ambiguities, are contrary.

(3). A final sign is superadded, which is necessarily contrary to the final sign of the original function.

Hence, taking the most unfavorable case, that in which all the ambiguities are of the same sign, we may by (2) take the upper signs without affecting the number of permanences; then the signs of the result, leaving out the last, are the same as those of the original function, and (3) this last introduces an additional variation. Thus there is at least one more variation than in the original.

Supposing, therefore, the product of all the factors corresponding to the negative and imaginary roots of an equation to have been formed, each multiplication by a factor corresponding to a positive root will introduce at least one change of sign; that is, an equation cannot have more positive roots than variations of sign.

To prove the second part of the rule, we have merely to put $-x$ for $x$ in the proposed equation; then, if the equation be complete, permanences are changed to variations, and *vice versa.* The transformed equation cannot have more positive roots than variations, that is, the proposed equation cannot have more negative roots than permanences.

**54.** Cor.—Hence, when the roots of an equation are all real, the number of variations is exactly equal to the number of positive roots, and the number of permanences to the number of negative roots. For if $m$ and $r$ be respectively the number of positive and negative roots, and $m'$ and $r'$ respectively the number of variations and permanences; then, since $m + r = m' + r'$, each being equal to the degree of the equation, and $m$ cannot exceed $m'$, nor $r$, $r'$, we must have $m = m'$, and $r = r'$.

**55.** This important theorem, generally called Descartes's Rule of Signs, from the name of its discoverer, is included as a particular case in Fourier's Theorem, (180), in connection with which some useful deductions will be given, which are usually deduced as corollaries from this proposition.

# CHAPTER IV.

## TRANSFORMATION OF EQUATIONS.

**56.** Without knowing the roots of an equation, it is in our power to derive from it various other equations the roots of which have given relations to those of the proposed equation. These transformations are, some, useful in preparing the equation for solution by reducing it to a more convenient form, and some are necessary to the actual solution. The propositions of the present chapter comprise, by no means all, but the most useful and simple cases of the general problem, to transform an equation into another the roots of which shall be any given functions of those of the proposed equation.

**57.** PROP. I.—*To transform an equation having negative or fractional exponents into another having only positive integral exponents.*

Let the proposed equation,

$$f(x) = ax^{-m} + bx^{r} + cx^{-s} + \dots . kx^{-t} + b = 0,$$

have negative exponents, integral or fractional. Suppose $-s$ to be the numerically greatest negative exponent; then, multiplying throughout by $x^s$, we have

$$f(x)x^s = ax^{s-m} + bx^{r+s} + c + \dots . kx^{s-t} + lx^s = 0,$$

in which no negative exponents occur. It is obvious that the roots of $f(x) = 0$ are not changed by this transformation, since, if $f(x) = 0$ for a certain value of $x$, then also $f(x)x^s = 0$ for the same value, and *vice versa.*

Again, to transform the equation,

$$f(x) = ax^{\frac{m}{n}} + bx^{\frac{p}{q}} + cx^{\frac{r}{s}} + \dots . kx^{\frac{t}{v}} + l = 0,$$

into another having only integral exponents, assume $y^{\mu} = x$,

where $\mu = nqs\ldots v$, or the L. C. M. of the denominators of the exponents. Substituting $y^\mu$ for $x$ in the proposed, we have

$$f(y) = ay^{m'} + by^{p'} + cy^{r'} + \ldots\ldots + ky^{t'} + l = 0,$$

where $m' = \frac{m}{n} \times \mu$, $p' = \frac{p}{q} \times \mu$, &c., are integers.

The relation between the roots of $f(y) = 0$ and $f(x = 0$ is $y = \sqrt[\mu]{x}$, that is, any root of the transformed equation in $y$ must be raised to the $\mu^{th}$ power to be a root of the proposed equation.

**58.** These results may be formulated in the following

RULE.—*To free an equation from negative exponents, multiply out by the reciprocal of whatever power of $x$ occurs with the numerically greatest negative exponent. To free an equation from fractional exponents, substitute in the proposed equation $y^\mu$ for $x$, where $\mu$ is the L. C. M. of the denominators of the exponents.*

Ex. I.—It is required to transform the equation,

$$3x^{-\frac{5}{2}} + 17x^{\frac{2}{3}} - 32x^{-4} + 9x - 8 = 0,$$

into another having only positive integral exponents.

Multiplying throughout by $x^4$, we have

$$3x^{\frac{3}{2}} + 17x^{\frac{14}{3}} - 32 + 9x^5 - 8x^4 = 0\,;$$

from this, by substituting $y^6$ for $x$, we obtain

$$3y^9 + 17y^{28} - 32 + 90y^{30} - 8y^{24} = 0,$$

or, $$9y^{30} + 17y^{28} - 8y^{24} + 3y^9 - 32 = 0.$$

### EXERCISES.

Transform the following equations into others having only positive integral exponents:

1. $x^2 + 3x^{-3} + 5x^{-4} - 7 = 0.$
2. $\frac{1}{x^2} + \frac{2}{x^3} - \frac{7}{x} - x^2 = 0.$

3. $3x - 5x^{\frac{2}{3}} + 7x^{\frac{5}{4}} - 7x^{-2} = 0.$

4. $2x^{\frac{7}{3}} + \frac{1}{2}x^{\frac{1}{2}} + 3x^{-3} = 0.$

5. $\dfrac{x^2 - 1}{1 - x^{\frac{1}{2}}} = 1 + x^{\frac{1}{3}}.$

6. $\sqrt{1 + x^3} = 3x^2 - x^{-3}.$

7. $\sqrt{1 - x^2} = \sqrt[3]{x - 1}.$

**59.** PROP. II.—*To transform an equation into another, whose roots shall be those of the proposed equation, each multiplied by a given quantity.*

Let $f(x) = C_n x^n + C_{n-1}x^{n-1} + \ldots . C_2x^2 + C_1x + C_0 = 0$

be an equation which we wish to transform into another, the roots of which shall be $m$ times as great as those of $f(x) = 0$. Suppose $y = mx$, then $\dfrac{y}{m} = x$. Replacing $x$ by $\dfrac{y}{m}$ in the proposed equation, we have

$$C_n\left(\frac{y}{m}\right)^n + C_{n-1}\left(\frac{y}{m}\right)^{n-1} + \ldots . + C_2\left(\frac{y}{m}\right)^2 + C_1\left(\frac{y}{m}\right) + C_0 = 0.$$

Multiplying throughout by $m^n$, we have

$$f(y) = C_n y^n + mC_{n-1}y^{n-1} + \ldots + m^{n-2}C_2y^2 + m^{n-1}C_1y + m^nC_0 = 0,$$

an equation whose roots are each $m$ times as great as those of $f(x) = 0$. Hence we have the following rule, before applying which we complete the equation by supplying the place of absent terms by zero's.

RULE.—*To transform an equation $f(x) = 0$ into another, $f(y) = 0$, the roots of which shall be $m$ times as great, replace $x$ by $y$, and multiply the coefficients, beginning with the second, by $m$, $m^2$, $m^3$, . . . . $m^n$.*

Ex. 1.—Transform $x^4 - 19x^3 + 49x^2 + 29x - 17 = 0$ into an equation having its roots three times as great.

Here $m = 3$; proceeding by the rule, we obtain

$$y^4 - 57y^3 + 441y^2 + 783y - 1377 = 0.$$

Ex. 2.—Transform $x^4 - 51x^2 + 176x - 789 = 0$ into an equation having its roots one-tenth as great.

First, completing the equation, we have,

$$x^4 + 0x^3 - 51x^2 + 176x - 789 = 0;$$

then proceeding by the rule, and multiplying by powers of $\frac{1}{10}$, we obtain

$$y^4 - .51y^2 + .176y - .0789 = 0.$$

**60.** The chief use of this transformation is to enable us to have unity as leading coefficient without introducing fractional coefficients; or, to clear the equation of fractional coefficients, without affecting the leading term with any coefficient other than unity.

Ex. 1.—Thus, if we have the equation

$$5x^4 - 7x^3 + 11x^2 + 5x - 3 = 0,$$

by taking $m = 5$, we have

$$5y^4 - 7 \times 5y^3 + 11 \times 5^2y^2 + 5 \times 5^3y - 3 \times 5^4 = 0,$$

or, $$y^4 - 7y^3 + 55y^2 + 125y - 375 = 0.$$

Ex. 2.—If we have to clear of fractional coefficients

$$x^4 + \frac{4}{3}x^3 - \frac{11}{25}x^2 + \frac{7}{60}x + \frac{13}{120} = 0,$$

we have to consider what $m$ must be, so that its powers will contain the denominators of the fractional coefficients.

By taking $m = 30$, and proceeding by the rule, we obtain

$$y^4 + \frac{4}{3}(30)\,y^3 - \frac{11}{25}(30)^2y^2 + \frac{7}{60}(30)^3y + \frac{13}{120}(30)^4 = 0,$$

or, $$y^4 + 40y^3 - 396y^2 + 3150y + 87750 = 0.$$

EXERCISES.

Transform the following equations into others having the leading coefficient unity, and the remaining coefficients integers:

1. $5x^3 - 3x^2 - 11x - 20 = 0.$
2. $7x^3 - 54x^2 + 30 = 0.$

3. $x^3 - 23.54x^2 + 36.7x - .745 = 0.$

4. $3x^4 + \frac{5}{7}x^2 - \frac{2}{5}x + \frac{5}{9} = 0.$

5. $x^5 - \frac{3}{11}x^2 - \frac{5}{2}x + 3 = 0.$

**61.** PROP. III.—*To transform an equation into another, whose roots are those of the proposed equation, with the contrary sign.*

Let $f(x) = C_n x^n + C_{n-1}x^{n-1} + \ldots . C_2 x^2 + C_1 x + C_0 = 0.$

The transformation of this equation into another, having roots numerically the same, but with contrary sign, is merely a particular case of the preceding transformation. We replace $x$ by $y$, and take $m = -1$, that is, multiply the coefficients, beginning at the second, by the successive powers of $-1$, which are alternately negative and positive; thus we obtain

$$C_n y^n - C_{n-1} y^{n-1} + \ldots . \pm C_2 y^2 \mp C_1 y \pm C_0 = 0$$

as the required equation.

RULE. — *The signs of the alternate terms of a complete equation being changed, the signs of all the roots will be changed.*

Ex. 1.—Let

$$x^5 + 11x^4 - 23x^3 - 47x^2 + 32x - 57 = 0;$$

then, replacing $x$ by $y$, and changing the signs of alternate terms, we have

$$y^5 - 11y^4 - 23y^3 + 47y^2 + 32y + 57 = 0,$$

an equation whose roots differ from those of the proposed only in having the contrary sign; *i. e.*, $y = -x$.

**62.** When an equation is incomplete, we may omit to complete it by supplying zero coefficients, remembering, when replacing $x$ by $y$, to change the signs of those terms containing odd or even powers of $x$, according as the exponent in the first term is even or odd. Thus,

Ex. 2.—Let $f(x) = x^6 + 34x^3 - 96x^2 + 5 = 0$, then
$$f(-x) = x^6 - 34x^3 - 96x^2 + 5 = 0.$$

The equation whose roots differ from those of $f(x) = 0$ in sign only, is often conveniently denoted, as above, by $f(-x) = 0$.

**63.** This simple transformation is very useful, as we are thus enabled to confine our attention to the discovery of rules for the determination of positive roots, it being always in our power to change negative into positive roots.

### EXERCISES.

Transform the following equations into others, whose roots are numerically the same, with contrary sign.

1. $x^3 - 17x^2 - 53x + 73 = 0.$
2. $x^4 - 23x^3 + 130x^2 - 305x + 96 = 0.$
3. $x^5 + 37x^4 - 111x^2 - 546 = 0.$
4. $6x^5 - 41x^3 - 65x^2 + 239x + 426 = 0.$
5. $8x^6 + 57x^3 - 172x^2 - 306x + 150 = 0.$
6. $x^8 - 73x^6 - 54x^4 + 29x^2 - 13 = 0.$

Note this last example, and explain why in this case $f(x)$ has the same signs as $f(-x)$.

**64.** PROP. IV.—*To transform an equation into another, whose roots are the reciprocals of those of the proposed equation.*

Let the proposed equation be

$$f(x) = C_n x^n + C_{n-1} x^{n-1} + \dots C_2 x^2 + C_1 x + C_0 = 0.$$

Assume $y = \frac{1}{x}$, and substitute $\frac{1}{y}$ for $x$ in $f(x)$; then

$$C_n\left(\frac{1}{y}\right)^n + C_{n-1}\left(\frac{1}{y}\right)^{n-1} + \dots C_2\left(\frac{1}{y}\right)^2 + C_1\left(\frac{1}{y}\right) + C_0 = 0.$$

Multiply throughout by $y^n$, and reverse the order of terms; then

$$C_0 y^n + C_1 y^{n-1} + C_2 y^{n-2} + \dots C_{n-1} y + C_n = 0,$$

an equation in which, since $y = \frac{1}{x}$, the roots are the reciprocals of those of the proposed equation.

RULE.—*The coefficients of the equation of the reciprocals of the roots are those of the proposed, written in reverse order.*

Ex. 1.—Find the equation involving the reciprocals of the roots of

$$5x^4 - 7x^3 + 11x^2 + 23x - 3 = 0.$$

Proceeding by the rule, and replacing $x$ by $y$, we have

$$-3y^4 + 23y^3 + 11y^2 - 7y + 5 = 0,$$

or,

$$3y^4 - 23y^3 - 11y^2 + 7y - 5 = 0,$$

which has for roots the reciprocals of the roots of the proposed.

**65.** This transformation is chiefly of use in Lagrange's Method of Solution, and in Budan's Method of Analyzing Equations, neither much used.

EXERCISES.

Find the equations having as roots the reciprocals of those of the following equations:

1. $3x^3 - 5x^2 + 20x - 11 = 0.$
2. $7x^4 - 54x^2 - 72x - 15 = 0.$
3. $x^5 - 17x^4 + 32x^2 + 7x - 23 = 0.$

**66.** PROP. V.—*To transform an equation into another, the roots of which shall be the squares of the roots of the proposed equation.*

Let $f(x) = C_n x^n + C_{n-1} x^{n-1} + \ldots . C_2 x^2 + C_1 x + C_0 = 0$;

the terms may be arranged in the following order,

$$f(x) = (C_n x^n + C_{n-2} x^{n-2} + \ldots . C_2 x^2 + C_0) \\ + (C_{n-1} x^{n-1} + C_{n-3} x^{n-3} + \ldots . C_3 x^3 + C_1 x) = 0,$$

the terms containing even powers of $x$ being collected, as here, with $C_n x^n$ when $n$ is even, otherwise the odd powers.

Similarly, since in $f(-x)$ the signs of the alternate terms are contrary to those of the corresponding terms in $f(x)$, we have

$$f(-x) = (C_n x^n + C_{n-2} x^{n-2} + \dots C_2 x^2 + C_0)$$
$$- (C_{n-1} x^{n-1} + C_{n-3} x^{n-2} + \dots C_3 x^3 + C_1 x) = 0,$$
$$\therefore\ f(x)\cdot f(-x) = (C_n x^n + C_{n-2} x^{n-2} + \dots C_2 x^a + C_0)^2$$
$$- (C_{n-1} x^{n-1} + C_{n-3} x^{n-3} + \dots C_3 x^3 + C_1 x)^2 = 0.$$

Squaring the expressions within the brackets, collecting, and arranging according to powers of $x$, we have for $f(x) f(-x)$

$$C_n^2 x^{2n} - (C_{n-1}^2 - 2C_n C_{n-2}) x^{2n-2} + (C_{n-2}^2 - 2C_{n-1} C_{n-3} + 2C_n C_{n-4}) x^{2n-4}$$
$$- (C_{n-3}^2 - 2C_{n-2} C_{n-4} + \&\text{c.}) x^{2n-6} + \dots \pm (C_1^2 - 2C_2 C_0) x^2$$
$$\mp C_0^2 = 0,$$

the coefficient of $(x^r)^2$ being

$$\pm (C_r^2 - 2C_{r-1} C_{r+1} + 2C_{r-2} C_{r+2} - 2C_{r-3} C_{r+3} + \dots \pm 2C_{r+m} \cdot C_{r-m}),$$

the upper sign being taken when $C_r$ stands in an odd term, and *vice versa,* and $C_{r+m}$ denoting the nearer of the extreme coefficients $C_n$ or $C_0$.

The equation thus obtained, by the product of $f(x)$ and $f(-x)$, and which we shall denote by $F(x^2) = 0$, has for roots the squares of the roots of $f(x) = 0$, since for every factor $(x - a)$ in $f(x)$, there is a corresponding factor $(x + a)$ in $f(-x)$, and therefore a factor $(x^2 - a^2)$ in $F(x^2)$; there are therefore $n$ such factors, and the equation $F(x^2) = 0$ has for roots, $a_1^2, a_2^2, a_3^2, \dots a_n^2$, if $a_1, a_2, a_3, \dots a_n$ are the roots of $f(x) = 0$.

We obtain the coefficients of $F(x^2)$ from those of $f(x)$ as follows:

RULE.—*Given a coefficient $C_r$ of $f(x) = 0$, the corresponding coefficient of $F(x^2) = 0$ may be found by taking from the square of $C_r$ the double product of the immediately contiguous coefficients, adding to the result the double product of the coefficients next removed on each side, and so on, as far as the coefficients extend; to the final result we prefix a positive sign, if $C_r$ stands in an odd term, the negative sign, if it stands in an even term.*

Ex.—Given, $x^4 - 5x^3 + 11x^2 + 7x - 6 = 0$; find the equation of the squares of the roots.

Here the expressions, which, with alternately positive and negative signs prefixed, will be the coefficients of $F(x^2)$, are,

$$1, \quad (25-22), \quad (121+70-12), \quad (49+132), \text{ and } 36;$$

the required equation is, therefore,

$$x^8 - 3x^6 + 179x^4 - 181x^2 + 36 = 0.$$

**67.** This transformation, as will be seen in Chap. X, is capable of being applied to good purpose in the analysis of equations.

EXERCISES.

Find the equations whose roots are the squares of the roots of the following equations:

1. $x^4 - 13x^3 + 12x^2 - 6x - 5 = 0.$
2. $5x^4 - 17x^2 - 23x + 19 = 0.$
3. $x^5 - 27x^3 + 90x^2 - 36x - 25 = 0.$

**68.** PROP. VI.—*To transform an equation into another, the roots of which are those of the proposed equation, each increased or diminished by a given quantity.*

Let $f(x) = C_n x^n + C_{n-1} x^{n-1} + \ldots . C_2 x^2 + C_1 x + C_0 = 0,$

be the given equation. Assume $y = x - a$, then whatever value is assigned to $x$, that of $y$ is less or greater according as $a$ is positive or negative. The required transformation is, therefore, $f(a+y) = 0$. By Art. 7 we have

$$f(a+y) = f(a) + f_1(a)y + \frac{1}{2}f_2(a)y^2 + \ldots \frac{1}{\underline{|n-1}}f_{n-1}(a)y^{n-1} + \frac{1}{\underline{|n}}f_n(a)y^n, = 0,$$

or, writing according to descending powers of $y$, and putting $C_n$ for $\frac{1}{\underline{|n}}f_n(a)$,

$$f(a+y) = C_n y^n + \frac{1}{\underline{|n-1}}f_{n-1}(a)y^{n-1} + \ldots + \frac{1}{2}f_2(a)y^2 + f_1(a)y + f(a) = 0,$$

an equation the roots of which are less than those of $f(x) = 0$ by the quantity $a$, if $a$ is positive, and *vice versa*.

**69.** In Art. 21 was explained and illustrated a convenient method for obtaining the value of $f(a)$. As intimated in (22), we can, by continuing the process with the successive quotients, obtain in succession the values of $f_1(a)$, $\frac{1}{2}f_2(a)$, $\frac{1}{\lfloor 3}f_3(a)$, &c., which are the coefficients of the transformed equation.

Ex. 1.—Transform the equation

$$5x^4 - 17x^3 + 29x^2 + 12x - 196 = 0$$

into another whose roots are each less by 3.

Writing down coefficients only, we proceed as follows:

$$\begin{array}{lrrrrrl}
 & 5 & -\ 17 & +\ 29 & +\ 12 & -196 & \underline{|\ 3\ \ } \\
 & & 15 & -\ \ 6 & +\ 69 & +243 & \\
\cline{2-6}
\text{1st Quot.} & 5 & -\ \ 2 & +\ 23 & +\ 81; & +\ \ 47 & = \text{1st Rem., or, } f(3). \\
 & & 15 & +\ 39 & +186 & & \\
\cline{2-5}
\text{2d Quot.} & 5 & +\ 13 & +\ 62; & +267 & & = \text{2d Rem., or, } f_1(3). \\
 & & 15 & +\ 84 & & & \\
\cline{2-4}
\text{3d Quot.} & 5 & +\ 28; & +146 & & & = \text{3d Rem., or, } \tfrac{1}{2}f_2(3). \\
 & & 15 & & & & \\
\cline{2-3}
\text{4th Quot.} & 5; & 43 & & & & = \text{4th Rem., or, } \tfrac{1}{\lfloor 3}f_3(3).
\end{array}$$

Writing out the transformed equation with these remainders as coefficients, we have

$$5y^4 + 43y^3 + 146y^2 + 267y + 47 = 0.$$

Since we have taken for $a$ a positive quantity 3, the roots of this transformed equation are each less than those of the proposed equation by 3. For the sake of exhibiting the entire quotient, we have taken down the leading coefficient each time in the above example; in practice the process is performed as in the following example:

Ex. 2.—Transform the equation

$$3x^5 + 19x^3 - 27x^2 - 121x + 20 = 0$$

into another whose roots shall be each less by 2.

Supplying the absent terms by zero, we proceed as follows:

$$
\begin{array}{rrrrrrl}
3 & +\,0 & +\,19 & -\,57 & -\,21 & +\,46 & \underline{|\,2} \\
 & \underline{6} & \underline{12} & \underline{62} & \underline{10} & \underline{-22} & \\
 & 6 & 31 & 5 & -11 & 24 & \\
 & \underline{6} & \underline{24} & \underline{110} & \underline{230} & & \\
 & 12 & 55 & 115 & 219 & & \\
 & \underline{6} & \underline{36} & \underline{182} & & & \\
 & 18 & 91 & 297 & & & \\
 & \underline{6} & \underline{48} & & & & \\
 & 24 & 139 & & & & \\
 & \underline{6} & & & & & \\
 & 30 & & & & &
\end{array}
$$

The transformed equation is,

$$3y^5 + 30y^4 + 139y^3 + 297y^2 + 219y + 24 = 0.$$

Ex. 3.—Transform the equation

$$8x^4 + 73x^3 + 147x^2 - 125x + 227 = 0$$

into another whose roots are each greater by 5.

Here we take $a = -5$, and proceed as before.

$$
\begin{array}{rrrrrl}
8 & +\,73 & +\,147 & -\,125 & +\,227 & \underline{|-5} \\
 & \underline{-40} & \underline{-165} & \underline{90} & \underline{+175} & \\
 & 33 & -18 & -35 & 402 & \\
 & \underline{-40} & \underline{35} & \underline{-85} & & \\
 & -7 & 17 & -120 & & \\
 & \underline{-40} & \underline{235} & & & \\
 & -47 & 252 & & & \\
 & \underline{-40} & & & & \\
 & -87 & & & &
\end{array}
$$

The transformed equation is,

$$8y^4 - 87y^3 + 252y^2 - 120y + 402 = 0.$$

**70.** This transformation may be regarded as the most important of those given in the present chapter. It will be found that both the analysis and solution of equations may be completely effected by means of a series of these transformations.

EXERCISES.

Transform the following equations into others whose roots are less, by the numbers placed to the right.

1. $x^4 - 15x^3 + 73x^2 + 712x - 85 = 0.$ $\qquad a = 5.$
2. $7x^4 - 46x^2 + 93x - 105 = 0.$ $\qquad a = 3.$
3. $11x^4 - 29x^3 + 56x + 7 = 0.$ $\qquad a = 4.$
4. $3x^5 - 17x^4 + 57x^3 - 78x^2 + 93 = 0.$ $\qquad a = -2.$
5. $8x^5 + 78x^2 - 107x - 540 = 0.$ $\qquad a = -5.$

**71.** Prop. VII.—*To transform an equation into another, in which any assigned term shall be absent.*

This is merely a particular case of the preceding transformation.

If the roots of $f(x) = 0$ be diminished by $k$, we have

$$f(k+y) = C_n y^n + \frac{1}{\underline{|n-1}} f_{n-1}(k)\, y^{n-1} + \ldots \frac{1}{\underline{|n-r}} f_{n-r}(k)\, y^{n-r} + \ldots f(k) = 0.$$

In order that the $(r+1)^{th}$ term in this may be absent, we must take $k$ so that the coefficient of $y^{n-r}$ may be zero, *i. e.*, $f_{n-r}(k) = 0$. From (7) we find that $f_{n-r}(k) = (C_n k^r + \frac{r}{n} C_{n-1} k^{r-1} + \frac{r(r-1)}{n(n-1)} C_{n-2} k^{r-2} + \text{\&c.}) \frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}}$, so that $k$ will have to be determined by an equation of the $r^{th}$ degree.

To take away the third term, $r$ becomes 2, and we find $k$ from the quadratic,

$$C_n k^2 + \frac{2}{n} C_{n-1} k + \frac{2}{n(n-1)} C_{n-2} = 0.$$

To take away the second term, a transformation that is often found advantageous, we determine $k$ from the simple equation $C_n k + \frac{1}{n} C_{n-1} = 0$, whence we obtain $k = -\frac{C_{n-1}}{nC_n}$.

Ex. 1.—Transform $x^3 - 15x^2 + 11x - 13 = 0$, into an equation without the second term.

Here $\frac{-C_{n-1}}{nC_n} = \frac{15}{3}, = 5$, and we proceed thus :

$$\begin{array}{rrrrrrrl}
1 & - & 15 & + & 11 & - & 13 & \underline{|\,5\quad} \\
 & & \underline{5} & & \underline{-50} & & \underline{-195} & \\
 & & -10 & & -39 & & -208 & \\
 & & \underline{5} & & \underline{-25} & & & \\
 & & -5 & & -64 & & & \\
 & & \underline{5} & & & & & \\
 & & 0 & & & & &
\end{array}$$

Therefore the required equation is,

$$y^3 - 64y - 208 = 0.$$

Ex. 2.—Transform $5x^3 - 11x^2 + 32x - 15 = 0$, into an equation without the second term.

In order to avoid fractional coefficients, we first by (59) transform the equation into another, the roots of which are 15 times as great; we thus obtain,

$$y^3 - 33y^2 + 1440y - 10125 = 0,$$

and proceed as before, obtaining,

$$z^3 + 1077z + 3053 = 0.$$

**72.** To take away any coefficient except the second is not of any practical advantage; it is besides evident that, in the great majority of cases, $k$ would be incommensurable, when determined by equations of the second and higher degrees.

### EXERCISES.

Deprive the following equations of their second terms:

1. $x^3 - 12x^2 + 15x - 3 = 0.$
2. $x^3 + 27x^2 - 32x - 54 = 0.$
3. $4x^3 - 2x^2 + 21x + 30 = 0.$
4. $x^4 - 24x^3 + 42x^2 - 37x + 49 = 0$
5. $3x^4 + 22x^3 - 56x + 13 = 0.$

# CHAPTER V.

## LIMITS OF THE ROOTS OF EQUATIONS.

**73.** In the preceding chapter have been investigated methods by which an equation may be reduced to a form convenient for solution. We have next to ascertain, as closely as possible, between what limits in the numerical scale the real roots lie, so as to avoid unnecessary labor in the search for those roots.

DEFINITION.—A *superior* limit to the positive roots of an equation is any number that is nearer to $+\infty$ than the greatest of these roots; an *inferior* limit, a number that is nearer to 0 than the least positive root.

A *superior* limit to the negative roots is any number nearer to 0 than the numerically least of these roots; an *inferior* limit, any number that is nearer to $-\infty$ than the numerically greatest of these roots.

A superior limit to the positive roots of $f(x)=0$, might also be defined as being such a number that, when it, or any greater number, is substituted for $x$ in $f(x)$, the result is positive: for if, for any substitution, we obtain a negative result, there must be a root greater than the number that produced that result.

In the following propositions respecting the limits of the roots of an equation, it is to be understood that the coefficient of the first term is unity, unless the contrary is stated.

**74.** PROP. I.—*The greatest negative coefficient of an equation, taken positively and increased by unity, is a superior limit to the positive roots.*

Let $-k$ be the numerically greatest negative coefficient in $f(x)=0$, which we shall suppose to be of the $n^{th}$ degree,

then any value of $x$ that makes $x^n - k(x^{n-1}+x^{n-2}+\ldots . x+1)$ positive, that is, $x^n > k\dfrac{x^n-1}{x-1}$, will, *a fortiori*, make $f(x)$ positive. For, even supposing that all the coefficients after the first are negative, none of them is, by supposition, numerically greater than $-k$. Now the inequality $x^n > k\dfrac{x^n-1}{x-1}$ is satisfied if $x^n =$ or $> x^n . \dfrac{k}{x-1}$, that is, if $x-1 =$ or $> k$, or, $x =$ or $> k+1$. Hence $f(x)$ is positive when $x = k+1$ or any greater number, that is, $k+1$ is a superior limit to the positive roots.

Ex. 1. $f(x) = x^4 - 15x^2 + 13x - 5 = 0.$

By the present proposition 15 is a superior limit to the positive roots.

**75.** PROP. II.—*In an equation of the $n^{th}$ degree, if $-k$ be the numerically greatest coefficient, and $x^{n-r}$ the highest power of $x$ that has a negative coefficient, then $\sqrt[r]{k}+1$ is a superior limit to the positive roots.*

Let $f(x) = 0$ be the proposed equation. Since all the terms that precede $x^{n-r}$ are, by supposition, positive, any value of $x$ that makes $x^n > k(x^{n-r} + x^{n-r-1} + \ldots . x + 1)$, that is, $x^n > k . \dfrac{x^{n-r+1}-1}{x-1}$ will obviously make $f(x)$ positive. The preceding inequality is satisfied if $x^n > k . \dfrac{x^{n-r+1}}{x-1}$, or if $x^{r-1}(x-1) > k$, or if $(x-1)^r =$ or $> k$, or if $x =$ or $> k^{\frac{1}{r}} + 1$. Hence for $x = \sqrt[r]{k}+1$, or any greater number, $f(x)$ is positive, that is, $\sqrt[r]{k}+1$ is a superior limit to the positive roots.

Ex.—Taking the same equation we had above,

$$x^4 - 15x^2 + 13x - 5 = 0,$$

we have $k = 15$, and $r = 4 - 2 = 2$; therefore $\sqrt{15}+1$, or 5, is a superior limit to the positive roots, a limit much closer than that obtained by the preceding method.

**76.** PROP. III.—*If each negative coefficient, taken positively, be divided by the sum of all the positive coefficients that precede it, the greatest quotient thus obtained will, increased by unity, be a superior limit to the positive roots.*

Let the proposed equation be,

$$f(x) = C_n x^n + C_{n-1} x^{n-1} + C_{n-2} x^{n-2} - C_{n-3} x^{n-3} + \dots . C_r x^r + \dots . C_0 = 0.$$

To demonstrate the proposition, we arrange $f(x)$ in a form deduced from the expression for the sum of a geometrical series, $\frac{x^m - 1}{x - 1} = x^{m-1} + x^{m-2} + \dots . x + 1$, whence we have,

$$x^m = (x-1)(x^{m-1} + x^{m-2} + \dots . x + 1) + 1.$$

We therefore transform all the positive terms of the equation by this formula, leaving the negative terms unchanged, thus:

$$\begin{array}{lllll}
C_n \; x^n & = C_n(x-1)x^{n-1} + & C_n \; (x-1)x^{n-2} + & C_n \; (x-1)x^{n-3} + \dots & C_n \\
+ C_{n-1} x^{n-1} = & & C_{n-1}(x-1)x^{n-2} + & C_{n-1}(x-1)x^{n-3} + \dots & C_{n-1} \\
+ C_{n-2} x^{n-2} = & & & C_{n-2}(x-1)x^{n-3} + \dots & C_{n-2} \\
- C_{n-3} x^{n-3} = & & & - C_{n-3} \qquad x^{n-3} &
\end{array}$$

and so on, with the remaining terms.

It is evident that in a given column only one negative coefficient can occur, since for different negative coefficients the powers of $x$ are different. If in any vertical column there occur no negative coefficient, that column, of course, is positive if $x$ is not less than unity. In a column, in which, as in that containing $x^{n-3}$ above, a negative coefficient occurs, we must, in order to obtain a positive result, have

$$(C_n + C_{n-1} + C_{n-2})(x - 1 > C_{n-3};$$

and

$$(C_n + C_{n-1} + \dots . C_{r-1})(x - 1 > C_r,$$

if the column containing $x^r$ has a negative coefficient $C_r$; that is, we must have $x$ greater than $\frac{C_{n-3}}{C_n + C_{n-1} + C_{n-2}} + 1$, and greater than $\frac{C_r}{C_n + C_{n-1} + \dots . C_{r-1}} + 1$.

If $x$, then, be taken greater than the greatest of these expressions, that value of $x$ will make all the vertical columns positive, and consequently make $f(x)$ positive. Hence the greatest of these expressions is a superior limit to the positive roots of $f(x) = 0$.

This is generally the most effective of the three limits yet given, and does not require the leading coefficient to be unity.

Ex. 1. $x^5 + 13x^4 - 21x^3 - 69x^2 + 81x - 58 = 0$.

By (74) we have $+ 70$ as a superior limit.

By (75) we have $\sqrt{69} + 1$, or 10, as a superior limit.

By the present proposition we take the greatest of the expressions, $\dfrac{21}{1+13} + 1$, $\dfrac{69}{1+13} + 1$, and $\dfrac{58}{1+13+81} + 1$; the second is plainly the greatest, therefore 6 is a superior limit.

Ex. 2. $3x^5 - 17x^4 + 56x^3 - 78x^2 + 93x - 105 = 0$.

By (74) we have 36 as a superior limit.

By (75) we obtain the same limit.

By this proposition we have $\dfrac{17}{3} + 1$, or 7, as superior limit.

Ex. 3. $x^5 + 7x^3 - 156x^2 - 453x - 954 = 0$.

By (74) we have 955 as superior limit.

By (75) we have $\sqrt[3]{954} + 1$, or 11, as superior limit.

**77.** It is not generally of much importance to obtain an inferior limit to the positive roots. Such a limit may be found by transforming the equation into another, whose roots are the reciprocals of those of the proposed equation. The reciprocal of a superior limit to the roots of this transformed equation, will be an inferior limit to the positive roots of the proposed equation.

Ex. 4. $x^3 + 15x^2 - 37x - 45 = 0$.

The equation of the reciprocals is,

$$45y^3 + 37y^2 - 15y - 1 = 0.$$

A superior limit to the positive roots of this is (76), $\frac{15}{45+37}+1$, or $\frac{97}{82}$; therefore $\frac{82}{97}$ is an inferior limit to the positive roots of the proposed equation.

**78.** To find limits to the negative roots, we change the alternate signs of the proposed equation (taking account of absent terms), thus obtaining the coefficients of the equation whose positive roots are numerically the same as the negative roots of the proposed equation. *Superior* and *inferior* limits to the positive roots of the transformed will, taken negatively, be *inferior* and *superior* limits to the negative roots of the proposed equation.

Ex. 5. $7x^4 - 13x^2 + 38x - 43 = 0.$

Merely taking the coefficients with alternate sign changed, we have $7 + 0 - 13 - 38 - 43$, from which we obtain $\frac{43}{7}+1$, or 8, as a superior limit. Hence $-8$ is an inferior limit to the negative roots of the proposed.

**79.** The limits obtained by the above methods are often far from close. Thus, in the preceding example, 8 was obtained as a superior limit to the positive roots of

$$7y^4 - 13y^2 - 38y - 43 = 0,$$

while 3 is, in reality, a superior limit. Among other methods that have been proposed, that of Newton gives the closest limit. As this method is, however, virtually comprised in the method of analysis given in (187), we refer to that article.

### EXERCISES.

By one of the preceding methods, find superior limits to the positive, and inferior limits to the negative roots of the following equations:

1. $x^3 + 15x + 36x - 54 = 0.$
2. $5x^4 - 36x^2 + 13x - 91 = 0.$
3. $2x^5 - 7x^4 + 9x^3 - 11x^2 + 13x - 15 = 0.$
4. $x^6 + 13x^3 - 39x^2 - 54x + 105 = 0.$

**80.** PROP. IV.—*If two numbers, successively substituted for the unknown quantity in an equation, give results with contrary signs, these numbers include between them an odd number of the roots of the equation; if the numbers give results with the same sign, they include either an even number of the roots, or no root.*

Let $p$ and $q$ be respectively *superior* and *inferior* limits to certain real roots $a_1$, $a_2$, $a_3$, .... $a_m$, of an equation $f(x) = 0$.

$$f(x) = (x-a_1)(x-a_2)(x-a_3)\dots(x-a_m)\,\phi(x), \quad [1]$$

where $\phi(x)$ is a function formed from the product of quadratic factors corresponding to imaginary roots, and binomial factors containing roots that lie without the limits $p$ and $q$, so that $\varphi(x)$ cannot change sign for any value of $x$ between $p$ and $q$.

If in [1] we put $p$ and $q$ successively for $x$, we have

$$f(p) = (p-a_1)(p-a_2)(p-a_3)\dots(p-a_m)\,\phi(p),$$
$$f(q) = (q-a_1)(q-a_2)(q-a_3)\dots(q-a_m)\,\phi(q).$$

Since all the factors $(p-a_1)$, $(p-a_2)$, .... $(p-a_m)$, are positive, and all the factors $(q-a_1)$, $(q-a_2)$, .... $(q-a_m)$, negative, while $\phi(p)$ and $\phi(q)$ have the same sign, $f(p)$ and $f(q)$ will have the same, or contrary signs, according as the number of the roots $a_1$, $a_2$, $a_3$, .... $a_m$, is even or odd. Hence, if $p$ and $q$ include between them an even number of roots, they produce results with the same sign, if an odd number, results with contrary signs.

Ex.
$$\begin{aligned} f(x) &= x^3 - 18x^2 + 101x - 180 = 0, \\ f(0) &= -180, \\ f(6) &= -6, \\ f(10) &= 30. \end{aligned}$$

As $f(0)$ and $f(6)$ have the same sign, we infer that there is either no root, or an even number of roots between 0 and 6; there are really two. As $f(6)$ and $f(10)$ have contrary signs, we infer the presence of some odd number of roots between 6 and 10.

**81.** It is obvious from this proposition that if we knew a quantity $k$ smaller than the least difference between any two unequal roots, we could, by substituting in succession $k$, $2k$, $3k$, &c., for $x$, obtain as many changes of sign as there are unequal roots in the equation, and thus ascertain their number and situation. For when, in the course of our substitutions, we should pass the least root, a change of sign would apprise us of the fact, and so on till we had passed all the unequal roots. In order to find such a quantity, less than the least difference of the roots, Waring proposed to transform the equation into another whose roots should be the squares of the differences of the roots. Then, supposing we found $l$ as an inferior limit to the positive roots of the transformed equation, $\sqrt{l}$ would be a quantity less than the least difference of the roots of the proposed equation, and we could employ it as above suggested. This method of separating the roots is, no doubt, theoretically perfect, but the great labor of forming the Equation of the Differences for equations above the fourth degree, has caused the method to be entirely abandoned.

**82.** PROP. V.—*If the first derived function $f_1(x)$, of an equation $f(x) = 0$, be equated to zero, a real root of $f_1(x) = 0$ lies between every adjacent two real roots of $f(x) = 0$.*

Let $a_1$, $a_2$, $a_3$, .... $a_m$, denote the real roots of $f(x) = 0$, arranged in descending order of magnitude, then

$$f(x) = (x-a_1)(x-a_2)(x-a_3)\ldots\ldots(x-a_m)\phi(x), \qquad [1]$$

where $\phi(x)$ is a function that cannot change sign, being the product of the quadratic factors corresponding to the imaginary roots, if any, of $f(x) = 0$.

In [1] we put $y + z$ for $x$; then, as $y + z - a = z + \overline{y - a}$,

$$f(y+z) = (z+\overline{y-a_1})(z+\overline{y-a_2})(z+\overline{y-a_3})\ldots(z+\overline{y-a_m})\phi(y+z).$$

If now the first member of this identity be expanded by Art. 7, and the second member by Art. 45, in ascending powers of $z$, we have

$$f(y)+f_1(y)z+\&c. = (S_m+S_{m-1}z+\ldots)(\phi(y)+\phi_1(y)z+\ldots),$$

where $S_m$ = the product of all the terms $\overline{y-a_1}, \overline{y-a_2}, \ldots \overline{y-a_m}$,
and $S_{m-1}$ = the sum of the products of the same terms, taken $m-1$ at a time.

Equating the coefficients of $z$, we have

$$f_1(y) = S_{m-1}\phi(y) + S_m\phi_1(y).$$

Substituting for $S_m$ and $S_{m-1}$ their values, and putting $x$ for $y$, as it is now immaterial what symbol we employ in this last result, we have

$$f_1(x) = \{(x-a_2)(x-a_3)\ldots(x-a_m)+(x-a_1)(x-a_3)\ldots(x-a_m) \\ +\ldots\}\phi(x)+(x-a_1)(x-a_2)(x-a_3)\ldots(x-a_m)\phi_1(x).$$

In this result we see: (1), Whichever of the quantities $a_1, a_2, \ldots a_m$, we put for $x$, the coefficient of $\phi_1(x)$ will vanish; (2), in the coefficient of $\phi(x)$, when we put for $x$ one of these quantities as $a_r$, all the products will vanish except that in which the factor $x-a_r$ is not found; (3), this product will determine the sign of $f(a_r)$, as $\phi(x)$ is always positive. Thus

$$\begin{aligned} f_1(a_1) &= (a_1-a_2)(a_1-a_3)\ldots\ldots(a_1-a_m)\phi(a_1), \\ f_1(a_2) &= (a_2-a_1)(a_2-a_3)\ldots\ldots(a_2-a_m)\phi(a_2), \\ f_1(a_3) &= (a_3-a_1)(a_3-a_2)\ldots\ldots(a_3-a_m)\phi(a_3), \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ f_1(a_m) &= (a_m-a_1)(a_m-a_2)\ldots\ldots(a_m-a_{m-1})\phi(a_m). \end{aligned}$$

These products are alternately positive and negative; for the first contains no negative factor, the second contains one, $(a_2-a_1)$, the third contains two, $(a_3-a_1)$ and $(a_3-a_2)$, the fourth contains three, and so on. Hence, by the preceding proposition, an odd number of the real roots of $f_1(x)=0$ must lie between every adjacent two of those of $f(x)=0$.

**83.** In the preceding article it has been assumed that the roots $a_1, a_2, a_3 \ldots a_m$, are all unequal. The conclusions there arrived at hold, however small the differences between

certain of the roots may be. Equal roots may be regarded as roots with a difference infinitely small, and the proposition holds with regard to them also, as may be seen from the following articles.

**84.** COR. 1.—*If* $f(x) = 0$ *have* $r$ *roots, each equal to* $a_1$, *then* $f_1(x) = 0$ *has* $r - 1$ *roots, each equal to* $a_1$.

For if $f(x) = 0$ has two roots each equal to $a_1$, then the factor $x - a_1$ must occur in each of the products that form the coefficient of $\phi(x)$, and consequently $f_1(a_1) = 0$. Thus, when $f(x)$ has two factors $x - a_1$, $f_1(x)$ has one such factor. If $f(x)$ has three factors $x - a_1$, $f_1(x)$ has two such factors; for after dividing each of these functions by $x - a_1$, the quotient from $f(x)$ will still have two factors $x - a_1$, and therefore the quotient from $f_1(x)$ must have one such factor. Thus, generally, when $f(x)$ has $r$ factors $x - a_1$, $f_1(x)$ has $r - 1$ such factors. That is, when $f(x) = 0$ has $r$ equal roots, $f_1(x) = 0$ has $r - 1$ equal roots, which we may conceive as lying one between each adjacent two of the $r$ equal roots of $f(x) = 0$, being indeed an arithmetical mean between these two.

The same observations apply to any other groups of equal roots in $f(x) = 0$. Suppose that $f(x) = 0$ has the root $a$ repeated $r$ times, the root $b$ repeated $s$ times, and the root $c$ repeated $t$ times, then $f(x)$ and $f_1(x)$ have $(x-a)^{r-1}(x-b)^{s-1}(x-c)^{t-1}$ as a common divisor.

**85.** COR. 2.—*Only one root of the equation* $f(x) = 0$ *can lie between any adjacent two of the roots of* $f_1(x) = 0$. For, if there could be two, there would be at least one root of $f_1(x) = 0$ lying between them, so that the roots of $f_1(x) = 0$, supposed to be adjacent, would not be adjacent.

**86.** COR. 3.—If $f(x) = 0$ has $m$ real roots, then $f_1(x) = 0$ must have at least $m - 1$ real roots in order to have one lying between each adjacent two of those of $f(x) = 0$. Hence, if an equation have all its roots real, the derived equation will also have all its roots real, each lying singly between a pair

of those of the proposed equation; the derived equation $f_1(x) = 0$ is in this case properly called the *limiting equation of* $f(x) = 0$.

**87.** Since $f_2(x)$ is, Art. 4, the first derived function of $f_1(x)$, $f_2(x) = 0$ will have an odd number of its roots lying between each adjacent pair of the real roots of $f_1(x) = 0$. If, then, $f(x) = 0$ has $m$ real roots, $f_1(x) = 0$ has at least $m - 1$ real roots, and $f_2(x) = 0$ has at least $m - 2$ real roots. Proceeding in this way, we arrive at the general statement that, *if* $f(x) = 0$ *has* $m$ *real roots,* $f_r(x) = 0$ *has at least* $m - r$ *real roots.*

**88.** Cor. 4.—If $f(x) = 0$, being of the $n^{th}$ degree, has $p$ imaginary roots, it has $n - p$ real roots; and since, by the preceding article, $f_r(x) = 0$ has at least $n - p - r$ real roots, it can, being of the $(n - r)^{th}$ degree, have only $p$ imaginary roots at most. Hence, *if any of its derived equations have* $p$ *imaginary roots,* $f(x) = 0$ *must have at least as many.*

**89.** Cor. 5.—If we know all the real roots of $f_1(x) = 0$, we can, by means of them, ascertain how many real roots $f(x) = 0$ contains.

Let $\alpha, \beta, \gamma, \ldots \kappa$ be the real roots of $f_1(x) = 0$, arranged in descending order of magnitude. We substitute these quantities, in order, for $x$ in $f(x)$, and note the signs of the results, $f(\alpha)$, $f(\beta)$, $f(\gamma), \ldots f(\kappa)$.

Then, according as $f(\alpha)$ is negative or positive, the proposed equation has, or has not, a root greater than $\alpha$. The equation has, or has not, a root between $\alpha$ and $\beta$, according as $f(\alpha)$ and $f(\beta)$ differ or agree in sign, and so on. Finally, if $f(\kappa)$ be positive for an equation of odd degree, or negative for one of even degree, there is a root of $f(x) = 0$ less than $\kappa$, otherwise not. The number of the real roots of $f(x) = 0$ is accordingly equal to the number of changes of sign in the series of results produced by the substitutions, in order, of $+\infty, \alpha, \beta, \gamma, \ldots \kappa, -\infty$ for $x$ in $f(x) = 0$.

**90.** This property of the roots of derived equations is the basis of a method suggested by Rolle for separating the real roots of an equation. He proposed, by means of the derived equation of the second degree, to find limits to the roots of the preceding derived equation of the third degree; thence limits to the roots of the antecedent derived equation of the fourth degree, and so on till limits were obtained to the roots of the proposed equation. This, called the Method of Cascades, has, like that of Waring, been entirely abandoned on account of the length of the calculations required.

NOTE.—An equation of the third degree can, of course, be easily analyzed in this manner, since its first derived function is of the second degree. It appears, however, to have been hitherto overlooked that an equation of the fourth degree can also be analyzed by the aid of a quadratic, or that, generally, any equation in which the second term from beginning or end is absent (a condition that can always be fulfilled by Art. 71), can be analyzed by the aid of an equation lower by two degrees. Let the equation be,

$$C_n x^n + 0 + C_{n-2} x^{n-2} + \ldots . \; C_2 x^2 + C_1 x + C_0 = 0. \qquad [1].$$

Forming (64) the equation of the reciprocals of this, and equating its first derived function to zero, we have

$$nC_0 y^{n-1} + (n-1)C_1 y^{n-2} + \ldots . \; 3C_{n-3} y^2 + 2C_{n-2} y = 0. \qquad [2].$$

An odd number of the roots of [2] lies between every adjacent two of the reciprocals of the real roots of [1]; therefore, taking the equation

$$2C_{n-2} z^{n-1} + 3C_{n-3} z^{n-2} + \ldots . \; (n-1)C_1 z^2 + nC_0 z = 0, \qquad [3]$$

whose roots are the reciprocals of those of [2], we have an equation whose roots separate those of [1]. One root of [3] is $z = 0$; the remaining roots can be determined by an equation of the degree $n - 2$. If we suppose [1] to be of the fifth degree, its roots can be separated by those of the cubic $2C_3 z^3 + 3C_2 z^2 + 4C_1 z + 5C_0 = 0$, and another root $z = 0$. If the equation be of the fourth degree, its roots will be separated by those of the quadratic $2C_2 z^2 + 3C_1 z + 4C_0 = 0$, which are given in the formula $z = \frac{-3C_1 \pm \sqrt{9C_1^2 - 32C_2C_0}}{4C_2}$, and another root $z = 0$.

Ex. 1. The equation $x^4 - 12x^2 + 12x - 3 = 0$ has for roots $x = -3.9..$, $x = .44..$, $.60..$, $2.8..$; by the above formula we obtain $z = 0$, $.5$, $1$, which separate the values of $x$.

Ex. 2. The equation $x^4 - 467x^2 + 3660x + 2826 = 0$ has for roots $x = 12.70828..$, $12.70820..$, $-.70..$, $-24.7..$; by the aid of the formula we obtain $z = 12.70824..$, $0$, $-.95..$, which separate the roots of the equation, two of which concur to six figures. We thus see that equations of less than the fifth degree can be easily analyzed by means of limiting equations.

Though an equation of the third degree in its most general form may be analyzed by means of a quadratic, it may be of use to note that when in the form $C_3 x^3 + C_1 x + C_0 = 0$, its two smaller roots, which are of the same sign as $C_0$, can be separated by $3C_0 \div -2C_1$, if they are real. For example, the equation $x^3 - 7x + 7 = 0$ has one root greater than $\frac{7 \times 3}{7 \times 2} = 1.5$, and one less. For exercises the student is referred to the numerous examples given on pages 125 and 153.

# CHAPTER VI.

## ON THE DEPRESSION OF EQUATIONS.

**91.** In the present chapter we shall treat of those equations which, on account of certain relations existing among the roots, are capable of depression, that is, of being resolved into equations of lower degrees.

Among the most important of these, are those equations which contain equal roots. We propose to show how these equal roots may be eliminated, and the solution of the equation containing them reduced to that of equations of lower degrees having only unequal roots.

**92.** Prop. I.—*An equation $f(x) = 0$ has, or has not, equal roots according as $f(x)$ and $f_1(x)$ have, or have not, a common factor involving $x$.*

As far as regards real roots, this has been shown in Art. 84, and that result may be made to include imaginary roots by a slight modification of the demonstration.

Thus, let $a_1, a_2, a_3, \ldots . a_n$ include all the roots of $f(x) = 0$, real or imaginary; then

$$f(x) = (x-a_1)(x-a_2)(x-a_3) \ldots . (x-a_n), \qquad [1].$$

$$f_1(x) = \{(x-a_2)(x-a_3)\ldots(x-a_n)+(x-a_1)(x-a_3)\ldots(x-a_n) + \&c. \} \quad [2].$$

It is evident that $f(x)$ and $f_1(x)$ have no common factor if all the factors in $f(x)$ are different; for then all the products in [2] are different, each being equal to $f(x)$ divided by a factor $(x-a_1)$, $(x-a_2)$, &c., different in each case; or,

$$f_1(x) = \frac{f(x)}{x-a_1} + \frac{f(x)}{x-a_2} + \frac{f(x)}{x-a_3} + \ldots . \frac{f(x)}{x-a_n}.$$

If in [1] we suppose $r$ of the roots to be each equal to $a$, $s$ of them equal to $b$, and $t$ of them equal to $c$, then

$$\begin{aligned} f_1(x) = \Big\{ & (x-a)^{r-1}(x-b)^s(x-c)^t \dots (x-a_n) + (x-a)^r(x-b)^{s-1} \\ & (x-c)^t \dots (x-a_n) + (x-a)^r(x-b)^s(x-c)^{t-1} \dots (x-a_n) \\ & + \text{products each containing } (x-a)^r(x-b)^s(x-c)^t \Big\}. \end{aligned}$$

Thus $(x-a)^{r-1}(x-b)^{s-1}(x-c)^{t-1}$ occurs as a factor in every term of $f_1(x)$, and is therefore a common factor of $f(x)$ and $f_1(x)$.

**93.** In order, therefore, to determine whether an equation $f(x) = 0$ has equal roots, we have only to ascertain whether $f(x)$ and $f_1(x)$ have a common divisor $\phi(x)$. If such a common measure be found, then the quotient of $f(x)$ by $\phi(x)$ will, equated to zero, contain all the roots of the proposed, without repetition. If $\phi(x)$ is of the form $(x-a)^r$, it will be found advantageous to divide by $(x-a)^{r+1}$, so as to obtain the quotient of as low degree as possible.

Ex. 1. Given $f(x) = x^4 - x^3 - 30x^2 + 32x + 160 = 0$; required to determine whether the equation has equal roots.

Here $f_1(x) = 4x^3 - 3x^2 - 60x + 32$, and we find a common measure $(x-4)$; $f(x)$, therefore, has a factor $(x-4)^2$, dividing by which, we find

$$f(x) = (x-4)^2(x^2+7x+10) = 0.$$

Thus the roots of the equation are, $-5$, $-2$, $4$, $4$.

Ex. 2. Given $f(x) = 4x^5 + 17x^4 + 8x^3 - 40x^2 - 32x + 16 = 0$; determine whether the equation has equal roots.

Here $f_1(x) = 20x^4 + 68x^3 + 24x^2 - 80x - 32$, and we find a common measure $x^2 + 4x + 4$; $f(x)$ has, therefore, a factor $(x+2)^3$. Dividing by this, we find

$$f(x) = (x+2)^3(4x^2-7x+2) = 0.$$

Thus the roots of the equation are, $-2$, $-2$, $-2$, and $\frac{1}{8}(7 \pm \sqrt{17})$.

**94.** The common measure of $f(x)$ and $f_1(x)$ may, however, be itself an expression containing more than one set of

repeated factors. We require, therefore, to deduce a systematic process for obtaining separate equations that contain only one set of factors.

Let $X_1$ be the product of all the factors that occur but once in $f(x)$; $X_2$, the product of the factors that occur twice; $X_3$, the product of those that occur three times; and so on, any of these factors, as $X_m$, being unity if there is no set of factors repeated $m$ times. Thus $f(x) = X_1 X_2^2 X_3^3 X_4^4 \ldots . X_r^r$.

Denoting the G. C. M. of $f(x)$ and $f_1(x)$ by $\phi_1(x)$, we have,

$$\phi_1(x) = X_2 X_3^2 X_4^3 \ldots . X_r^{r-1},$$

and denoting the G. C. M. of $\phi_1(x)$ and its derived function $\phi_1'(x)$ by $\phi_2(x)$, we have,

$$\phi_2(x) = X_3 X_4^2 \ldots . X_r^{r-2}.$$

In like manner we obtain in succession,

$$\begin{aligned} \phi_3(x) &= X_4 X_5^2 \ldots . X_r^{r-3}, \\ \phi_4(x) &= X_5 \ldots . X_r^{r-4}, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ \phi_{r-1}(x) &= X_r. \\ \phi_r(x) &= 1, \text{ since } X_r \text{ has only unequal factors.} \end{aligned}$$

From these functions we obtain by division,

$$\begin{aligned} F_1(x) &= \frac{f(x)}{\phi_1(x)} = X_1 X_2 X_3 \ldots . X_r, \\ F_2(x) &= \frac{\phi_1(x)}{\phi_2(x)} = X_2 X_3 \ldots . X_r, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ F_{r-1}(x) &= \frac{\phi_{r-2}(x)}{\phi_{r-1}(x)} = X_{r-1} X_r, \\ F_r(x) &= \frac{\phi_{r-1}(x)}{\phi_r(x)} = X_r. \end{aligned}$$

We can now obtain the separate factors $X_1, X_2, \ldots . X_{r-1}$, by another division; thus,

$$\frac{F_1(x)}{F_2(x)} = X_1; \quad \frac{F_2(x)}{F_3(x)} = X_2; \ldots . \frac{F_{r-1}(x)}{F_r(x)} = X_{r-1}.$$

Finally, by equating the functions $X_1, X_2, \ldots . X_r$, to zero, we obtain the roots of the proposed equation; any root that occurs in $X_m$, for example, occurring $m$ times in $f(x) = 0$.

The process may be presented as follows:

$$\begin{array}{lllll} f\,(x), & \phi_1(x), & \phi_2(x), & \ldots\ldots & \phi_{r-1}(x), \\ F_1(x), & F_2(x), & F_3(x), & \ldots\ldots & F_r\,(x), \\ X_1\quad, & X_2\quad, & X_3\quad, & \ldots\ldots & X_r\quad, \end{array}$$

In the first line, each term, after the first, is the G. C. M. of the preceding term and its first derived function. In the second line, each term is the quotient of the term under which it stands, by the next term in the same line. In the third line, each term is the quotient of the term under which it stands by the next term in the second line.

The following example will illustrate the process.

$$\begin{aligned} f\,(x) &= x^8 - 7x^7 - 2x^6 + 118x^5 - 259x^4 - 83x^3 + 612x^2 \\ &\qquad - 108x - 432 = 0. \\ \phi_1(x) &= x^4 - 7x^3 + 13x^2 + 3x - 18. \\ \phi_2(x) &= x - 3. \\ \phi_3(x) &= 1. \\ F_1(x) &= x^4 - 15x^3 + 10x + 24. \\ F_2(x) &= x^3 - 4x^2 + x + 6. \\ F_3(x) &= x - 3. \\ X_1 &= x + 4. \\ X_2 &= x^2 - x - 2. \\ X_3 &= x - 3. \\ \therefore\quad f\,(x) &= (x+4)(x^2-x-2)^2(x-3)^3. \end{aligned}$$

**95.** COR.—When the coefficients of $f(x)$ are all commensurable, the functions $X_1, X_2, \ldots . X_r$, have likewise all their coefficients commensurable. Hence

(1). If only one of the roots of $f(x) = 0$ is repeated $r$ times, that root must be commensurable; for it will be determined by an equation $X_r = 0$, which will be of the first degree, and contains no incommensurable quantities.

(2). Hence incommensurable equal roots will be determined by equations of at least the second degree; and any equation that contains such roots must have a factor $X_m{}^m$,

where $m$ cannot be less than two; that is, the equation must, at least, have a factor $X_2^2$ of the fourth degree. Hence

(3). An equation of the third degree with commensurable coefficients cannot have equal roots that are not commensurable; and

(4). An equation of the fourth degree with commensurable coefficients cannot have incommensurable equal roots, unless its first member is a perfect square;

(5). An equation of the fifth degree cannot have incommensurable equal roots, unless it has one commensurable root.

(6). Equations of the sixth and higher degrees may have such roots, even when their coefficients are all commensurable.

**96.** We are thus in possession of a method by which we can eliminate all equal roots from an equation, and obtain other equations of inferior degrees involving roots without any repetition. The process, however, though simple in theory, involves much numerical labor in equations of high degree, with large coefficients, as will be fully appreciated when we come to exemplify Sturm's Method, which consists essentially in an operation similar to that of finding the G. C. M. of a proposed function, and its first derived function.

### EXERCISES.

Find the equal roots in the following equations:

1. $x^3 - x^2 - 8x + 12 = 0.$
2. $4x^3 + 12x^2 - 63x + 54 = 0.$
3. $x^4 - 72x^2 + 220x + 75 = 0.$
4. $8x^4 - 52x^3 + 66x^2 + 77x - 49 = 0.$
5. $x^5 + x^4 - 18x^3 + 14x^2 + 21x + 5 = 0.$
6. $x^6 - 13x^4 + 12x^3 + 97x^2 + 66x + 12 = 0.$

## RECIPROCAL EQUATIONS.

**97.** In Art. 64 we found that an equation

$$x^n + C_{n-1}x^{n-1} + C_{n-2}x^{n-2} + \dots C_2x^2 + C_1x + C_0 = 0$$

may be transformed into an equation whose roots are the

reciprocals of those of the proposed, by simply writing the coefficients in reverse order, and replacing $x$ by some other symbol. If the resulting transformed equation be identical with the proposed equation, it is obvious that for every value $a$ that satisfies either equation, there must be another value $\frac{1}{a}$ that also satisfies it. Such equations are called reciprocal equations, and it will be shown that they can be depressed to equations of one-half their own degree.

**98.** To find what relations must exist among the coefficients in order that a proposed equation may be a reciprocal equation, suppose the equation above to be transformed by the rule in (64), and divide through by $C_0$; thus we obtain

$$y^n + \frac{C_1}{C_0}y^{n-1} + \frac{C_2}{C_0}y^{n-2} + \dots \frac{C_{n-2}}{C_0}y^2 + \frac{C_{n-1}}{C_0} + 1 = 0.$$

In order that the transformed may be identical with the proposed equation we must have

$$\frac{C_1}{C_0} = C_{n-1}, \text{ or } C_1 = C_0C_{n-1}; \; C_2 = C_0C_{n-2}; \dots C_r = C_0C_{n-r}; \; 1 = C_0^2.$$

From the last equation we obtain $C_0 = +1$, or $C_0 = -1$, and this gives rise to two classes of reciprocal equations.

*First.* If $C_0 = +1$, we have

$$C_1 = C_{n-1}; \quad C_2 = C_{n-2}; \dots \; C_r = C_{n-r}; \dots \text{\&c.}$$

Therefore *an equation is a reciprocal equation when coefficients equidistant from the first and last are equal.*

*Secondly.* If $C_0 = -1$, we have

$$C_1 = -C_{n-1}; \quad C_2 = -C_{n-2}; \dots \; C_r = -C_{n-r}; \text{ \&c.}$$

In this case, supposing $n = 2m$, we should have the middle term $C_m = -C_m$, which is impossible unless $C_m = 0$.

Therefore *an equation is a reciprocal equation when coefficients equidistant from the first and last are numerically*

*equal though of contrary signs, provided that the middle coefficient be zero, if the equation be of even degree.*

Ex. 1. $x^6 - 7x^5 + 45x^4 + 12x^3 + 45x^2 - 7x + 1 = 0$, is a reciprocal equation of the first class.

Ex. 2. $x^8 + 11x^7 - 5x^6 - 31x^5 + 31x^3 + 5x^2 - 11x - 1 = 0$, is a reciprocal equation of the second class.

**99.** A reciprocal equation of the first class, and of odd degree, is obviously satisfied by $x = -1$, and has, therefore, its first member divisible by $x + 1$. Since the remaining roots occur in reciprocal pairs, the quotient arising from this division will, equated to zero, be a reciprocal equation of even degree with its final term positive.

In like manner, a reciprocal equation of the second class, and of odd degree, has its first member divisible by $x - 1$, and can thus be depressed to a reciprocal equation of even degree with its final term positive.

A reciprocal equation of the second class, and of even degree, is evidently satisfied by both $x = 1$, and $x = -1$; its first member is therefore divisible by $x^2 - 1$, and will furnish a quotient which, equated to zero, is a reciprocal equation of even degree with its final sign positive.

Every reciprocal equation is therefore either of even degree with its final sign positive, or may be reduced to that form preparatory to depression by the following theorem

**100.** Prop. I.—*A reciprocal equation of even degree with its last term positive, may be depressed to an equation of one-half the degree of the proposed.*

Let $x^{2m} + C_1 x^{2m-1} + C_2 x^{2m-2} + \ldots C_2 x^2 + C_1 x + 1 = 0$ [1]

be the proposed equation.

Collecting in pairs the terms that are equidistant from the first and last, and dividing by $x^m$, we have

$$\left(x^m + \frac{1}{x^m}\right) + C_1\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + C_2\left(x^{m-2} + \frac{1}{x^{m-2}}\right) + \&c. = 0. \quad [2].$$

Let

$$x + \frac{1}{x} = y; \quad \text{then}$$

$$x^2 + \frac{1}{x^2} = y^2 - 2,$$

$$x^3 + \frac{1}{x^3} = \left(x^2 + \frac{1}{x^2}\right)\left(x + \frac{1}{x}\right) - y \qquad = y^3 - 3y,$$

$$x^4 + \frac{1}{x^4} = \left(x^3 + \frac{1}{x^3}\right)\left(x + \frac{1}{x}\right) - \left(x^2 + \frac{1}{x^2}\right) = y^4 - 4y^2 + 2,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\left(x^m + \frac{1}{x^m}\right) = \left(x^{m-1} + \frac{1}{x^{m-1}}\right)\left(x + \frac{1}{x}\right) - \left(x^{m-2} + \frac{1}{x^{m-2}}\right) = y^m - my^{m-2} + \&c.$$

By substituting these values in [2], we obtain an equation of the $m^{th}$ degree in $y$, that is, an equation of half the degree of the proposed. Any root, as $a$, of this equation will give two roots of the proposed by means of the relation $x + \dfrac{1}{x} = a$; or, $x^2 - ax + 1 = 0$.

Ex. Let $2x^6 - 12x^5 + 19x^4 - 19x^2 + 12x - 2 = 0$.

Since both $+1$ and $-1$ are roots by inspection, we divide the left hand member by $x^2 - 1$, and obtain,

$$2x^4 - 12x^3 + 21x^2 - 12x + 2 = 0,$$

therefore $$2\left(x^2 + \frac{1}{x^2}\right) - 12\left(x + \frac{1}{x}\right) + 21 = 0,$$

and putting $x + \dfrac{1}{x} = y$, we obtain

$$2(y^2 - 2) - 12y + 21 = 0,$$

or $$2y^2 - 12y + 17 = 0,$$

thence $$y = \tfrac{1}{2}(6 \pm \sqrt{2}),$$

and $$x = 2 \pm \sqrt{2} \text{ or } \tfrac{1}{2}(2 \pm \sqrt{2}.$$

## EXERCISES.

Solve the following reciprocal equations:

1. $3x^4 - 7x^3 + 31x^2 - 7x + 3 = 0$.
2. $5x^4 + 8x^3 - 56x^2 + 8x + 5 = 0$.

3. $x^5 + 14x^4 - 25x^3 - 25x^2 + 14x + 1 = 0.$
4. $8x^5 - 52x^4 - 79x^3 + 79x^2 + 52x - 8 = 0.$
5. $x^6 - 23x^5 - 84x^4 + 84x^2 + 23x - 1 = 0.$

## BINOMIAL EQUATIONS.

**101.** These equations are such as consist of two terms only, as,

$$x^n - A = 0,$$

where $A$ is a known quantity. This is the only extensive class of equations that are capable of complete solution by a general method. Their general solution, however, depends on De Moivre's formula, and is contained in works on Trigonometry. The following propositions comprise the theory of Binomial Equations and the solution of such cases as are readily solvable by algebraical processes.

**102.** PROP. I.—*The roots of a binomial equation are all different.*

For the first derived function of $x^n - A$ is $nx^{n-1}$, and no value of $x$ can make $x^n - A$ and $nx^{n-1}$ vanish simultaneously. See Art. 92.

**103.** COR.—Any algebraical quantity (that is, any quantity included under the general form $a + b\sqrt{-1}$, where $a$ and $b$ are real quantities, positive, negative, or zero), has $n$ different $n^{th}$ roots. From the above equation we have $x = \sqrt[n]{A}$, and $x$ has $n$ different values (36).

**104.** PROP. II.—*All the $n^{th}$ roots of an algebraical quantity may be found by multiplying one of them by the $n$ different $n^{th}$ roots of unity.*

For in the equation $x^n - A = 0$ put $ay$ for $x$, where $a$ denotes one of the $n^{th}$ roots of $A$. Then we have

$$a^n y^n = a^n; \text{ or, } y^n = 1.$$

From this we have $y = \sqrt[n]{1}$. Hence $x = \sqrt[n]{A} = ay = a\sqrt[n]{1}$.

**105.** Since we can always, as in the preceding article, reduce a binomial equation to the form $x^n \pm 1 = 0$, we shall, in the following propositions, confine our attention to this form.

**106.** PROP. III.—*If* $a$ *be any root of the equation* $x^n - 1 = 0$, *then any integral power of* $a$ *will also be a root.*

For $$(a^m)^n = a^{mn} = (a^n)^m = 1^m = 1.$$

**107.** COR.—It hence appears that the roots of the equation $x^n - 1 = 0$ may be represented under an infinite variety of forms, since each term of the series $a^{-\infty}, \ldots . a^{-3},\ a^{-2},\ a^{-1},\ a^0,\ a,\ a^2,\ a^3, \ldots .$ is a root. Of these there cannot, however, be more than $n$ essentially different, as otherwise the equation would have more than $n$ roots.

**108.** PROP. IV.—*If* $a$ *be any root of* $x^n + 1 = 0$, *then any odd integral power of* $a$ *will also be a root.*

For $(a^m)^n = (a^n)^m = (-1)^m = -1$, if $m$ is odd.

**109.** PROP. V.—*If* $m$ *be prime to* $n$, *the equations* $x^m - 1 = 0$, *and* $x^n - 1 = 0$, *have no common root but unity.*

Let $p$ and $q$ be integers such that $pm - qn = 1$,* and suppose $a$ is a root common to the two equations. Then, as $a^m = 1$, and $a^n = 1$, we have also $a^{pm} = 1$, and $a^{qn} = 1$ (105). Hence, by division, $a^{pm-qn} = 1$, or $a^1 = 1$; that is, the only common root is unity.

**110.** PROP. VI.—*If* $n$ *is a prime number, and* $a$ *an imaginary root of* $x^n - 1 = 0$, *then all the roots of the equation are found in the series,* $1,\ a,\ a^2,\ a^3, \ldots .\ a^{n-1}$.

For these are all roots by Prop. V; and no two of them are equal. For, if possible, let $a^p = a^q$, then $a^{p-q} = 1$, that is, $a$ is a root of $x^{p-q} - 1 = 0$, and also of $x^n - 1 = 0$, which is impossible, since $p - q$ being less than $n$ must be prime to it.

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* Such integers can always be found by algebra.

**111.** PROP. VII.—*The solution of* $x^n - 1 = 0$, *where* $n$ *is a composite number, may be made to depend upon the solution of the equations* $x^p - 1 = 0$, $x^q - 1 = 0$, *&c., where* $p$, $q$, $r$, *&c., are the different prime factors of* $n$.

First suppose $n = pq$, where $p$ and $q$ are prime to each other. Then $x^n - 1 = 0$, or $x^{pq} - 1 = 0$, is divisible by both $x^p - 1$ and $x^q - 1$. By Prop. VI, the roots of $x^p - 1 = 0$, are,

$$1,\ a,\ a^2,\ a^3, \dots\ a^{p-1},$$

and those of $x^q - 1 = 0$, let us suppose are,

$$1,\ \beta,\ \beta^2,\ \beta^3, \dots\ \beta^{q-1},$$

and all these are roots of $x^n - 1 = 0$. Also the products formed by multiplying each term in the first row by each term in the second are all roots. For each of these products is of the form $a^r\beta^s$, and since $a^{rn} = 1$, and $\beta^{sn} = 1$, therefore $(a^r\beta^s)^n = 1$, that is, $a^r\beta^s$ is a root. Moreover, no two of these products are alike. For, if possible, suppose $a^r\beta^s = a^t\beta^v$, then $a^{r-t} = \beta^{v-s}$. But as $a^{r-t}$ is a root of $x^p - 1 = 0$, and $\beta^{v-s}$ a root of $x^q - 1 = 0$, these equations have a common root besides unity, which (110) is impossible, since $p$ and $q$ are prime to each other. Therefore the $pq$ products, formed by multiplying each root of $x^p - 1 = 0$ by each root of $x^q - 1 = 0$, are the roots of $x^{pq} - 1 = 0$, or $x^n - 1 = 0$.

In the same way, if $n$ be the product of three prime factors $p$, $q$, $r$, it may be proved that the roots of $x^n - 1 = 0$ are the $pqr$ products obtained by multiplying together the roots of the equations $x^p - 1 = 0$, $x^q - 1 = 0$, $x^r - 1 = 0$; and similarly for any number of prime factors.

**112.** Again, let $n$ be a power of some number, as $n = p^2$. Suppose the roots of $x^p - 1 = 0$ are

$$1,\ a,\ a^2,\ a^3, \dots\ a^{p-1},$$

then these as well as

$$1,\ \sqrt[p]{a},\ \sqrt[p]{a^2},\ \sqrt[p]{a^3}, \dots\ \sqrt[p]{a^{p-1}}$$

are roots of $x^{p^2} - 1 = 0$, as is also the product of each root in the first row by each root in the second. We have there-

fore $p^2$ different quantities of the general form $a^r \cdot \sqrt[p]{a^s}$ all satisfying the proposed equation, and these are therefore all the roots.

But by the solution of $x^p - 1 = 0$ we obtain only the $p$ roots given in the first row above; those in the second row must obviously be determined by solving the proposed equation. A similar remark applies when $n = p^r$, where $r = 3$, or any greater number.

Thus we can find all the roots of $x^{15} - 1 = 0$ by solving $x^5 - 1 = 0$, and $x^3 - 1 = 0$; but if $x^{25} - 1 = 0$ be given, we can find by this method only five roots, and if $x^{49} - 1 = 0$ be given, only seven roots.

**113.** Prop. VIII.—*The solution of an equation* $x^n \pm 1 = 0$ *may be reduced to that of an equation of not more than one-half the degree of the proposed equation, and having all its roots real.*

For, taking the most unfavorable case, $x^n + 1 = 0$, where $n$ is an even number, and the equation has therefore no real root, we see that this is a reciprocal equation of even degree with its final term positive; it may therefore (100) be depressed to an equation in $y$ of one-half the degree of $x^n + 1 = 0$. As $y = x + \dfrac{1}{x}$, or $x^2 - yx + 1 = 0$, $y$, being the sum of a pair of conjugate roots, is real.

$x^n - 1 = 0$, when $n$ is even may be depressed two degrees by dividing by $x^2 - 1$; and $x^n \pm 1 = 0$, when $n$ is odd, may be depressed one degree by dividing by $x \pm 1$. In either case the depressed equations are reciprocal equations of even degree with their final terms positive, and these again may be depressed to equations in $y$ of less than one-half the degree of the proposed equations.

Ex. 1.
$$x^3 - 1 = 0,$$
$$\therefore \quad (x-1)(x^2+x+1) = 0.$$

Hence the roots are 1, and $-\frac{1}{2}(1 \pm \sqrt{-3})$, which values are the cube roots of unity. By changing their signs, we obtain the three cube roots of $-1$.

Ex. 2. $$x^4 + 1 = 0.$$

Putting $y = x + \dfrac{1}{x}$, we obtain $y^2 - 2 = 0$, whence $y = \pm \sqrt{2}$,

and $\quad x^4 + 1 = (x^2 + \sqrt{2} \cdot x + 1)(x^2 - \sqrt{2} \cdot x + 1)$,

from which we obtain as the four fourth roots of unity,

$$\tfrac{1}{2}(\sqrt{2} \pm \sqrt{-2}), \text{ and } -\tfrac{1}{2}(\sqrt{2} \pm \sqrt{-2}).$$

Ex. 3. $$x^5 - 1 = 0.$$

From this, dividing by $x - 1$, we obtain

$$x^4 + x^3 + x^2 + x + 1 = 0.$$

Putting $y = x + \dfrac{1}{x}$, we have

$$y^2 + y - 1 = 0\,; \text{ whence } y = -\tfrac{1}{2}(1 \pm \sqrt{5}).$$

Therefore

$$x^5 - 1 = (x-1)\left(x^2 + \frac{1+\sqrt{5}}{2} \cdot x + 1\right)\left(x^2 + \frac{1-\sqrt{5}}{2} \cdot x + 1\right),$$

whence we obtain as the five fifth roots of unity,

$$1,\ \tfrac{1}{4}\left(\sqrt{5}-1 \pm \sqrt{-10-2\sqrt{5}}\right),\ -\tfrac{1}{4}\left(\sqrt{5}+1 \pm \sqrt{-10+2\sqrt{5}}\right).$$

These, with changed signs, are the fifth roots of $-1$.

Ex. 4. $$x^6 - 1 = 0.$$

By Prop. V, the solution will be effected by finding the roots of $x^3 - 1 = 0$, which are 1 and $-\frac{1}{2}(1 \pm \sqrt{-3})$, and those of $x^2 - 1 = 0$, which are 1 and $-1$; the products of these sets of roots will furnish the six sixth roots of unity.

**114.** When we proceed in the same way to solve $x^7 - 1 = 0$, we obtain a reduced equation in $y$ of the third degree; for $x^9 - 1 = 0$, a reduced equation in $y$ of the fourth degree, and so on.

These algebraical methods of obtaining the roots of binomial equations are of little practical importance, since, as before mentioned, the roots of such equations are most readily

and symmetrically obtained by the aid of De Moivre's Formula.

**115.** Besides the three preceding classes of equations, any equation is capable of depression when we know that the roots, or some of them, are known functions of each other. If the roots, for example, are known to be in geometrical, or arithmetical progression, we can by means of the known relations between the roots and coefficients, Art. 45, easily depress the proposed equations to others of lower degree.

In general, if the relation $a_2 = \phi(a_1)$ is known to exist among two of the roots, $a_1$, and $a_2$, of $f(x) = 0$, we may depress the equation as follows:

Substitute $\phi(x)$ for $x$ in $f(x)$, and let $F(x)$ be the resulting function. Then we have both $f(x) = 0$, and $F(x) = 0$, when $x = a_1$. For $a_1$ is a root of $f(x) = 0$ by supposition, and to put $a_1$ for $x$ in $F(x)$ is equivalent to putting $a_2$ for $x$ in $f(x)$. Therefore $f(x)$ and $F(x)$ have a common factor $x - a_1$ which can be found. Then $a_1$ being known, $a_2 = \phi(a_1)$ becomes known, and the equation may be depressed two dimensions.

Ex. Suppose it is known that two roots $a_1$ and $a_2$ of

$$x^4 - 9x^3 + 7x^2 + 6x + 16 = 0, \qquad [1]$$

are such that $a_2 = 3a_1 + 2$.

Substitute $3x + 2$ for $x$ in the equation, then

$$(3x+2)^4 - 9(3x+2)^3 + 7(3x+2) + 6(3x+2) + 16 = 0,$$

or, $$81x^4 - 27x^3 - 207x^2 - 126x + 0 = 0,$$

or, $$9x^3 - 3x^2 - 23x - 14 = 0. \qquad [2].$$

We find $(x-2)$ as the G. C. M. of [1] and [2]. Therefore $a_1 = 2$, and $a_2 = 3a_1 + 2 = 8$. Thus we find

$$x^4 - 9x^3 + 7x^2 + 6x + 16 = (x-2)(x-8)(x^2+x+1),$$

and the depressed equation is

$$x^2 + x + 1 = 0.$$

**116.** Whenever in the course of the analysis of an equation we find a commensurable root $a_1$, it is advisable to depress the equation by dividing by $(x-a_1)$, since we thus greatly facilitate the finding of the remaining roots. As these commensurable roots will present themselves in the course of the preliminary analysis recommended in Chap. X, we consider it unnecessary to make a special research for such roots, which are of comparatively rare occurrence.

### EXERCISES.

Solve the following equations:

1. $x^3 - 5 = 0.$
2. $x^4 + 2 = 0.$
3. $x^5 - 7 = 0.$
4. $x^6 + 4 = 0.$
5. $x^{10} - 1 = 0.$
6. $x^{10} + 3 = 0.$
7. $x^{15} + 1 = 0.$
8. $x^{15} - 20 = 0.$

Depress the following equations:

1. $x^7 - 1 = 0.$
2. $x^9 + 1 = 0.$
3. $x^{11} + 1 = 0.$
4. $x^{12} - 1 = 0.$

# CHAPTER VII.

## SOLUTION OF EQUATIONS BY GENERAL FORMULAS.

**117.** The only equations for which general solutions have been found are those of the first four degrees, and the equations that can be depressed to any of those degrees.

Leaving aside equations of the first and second degrees, which are fully treated of in elementary algebra, we propose in the present chapter to give the algebraical solution of cubic and biquadratic equations.

### CUBIC EQUATIONS.

**118.** By the method given in Art. 71, any proposed cubic may be reduced to the form,

$$x^3 + qx + r = 0. \qquad [1].$$

Assume that $x$ is the sum of two other unknown quantities, that is, $x = y + z$;
then $x^3 = 3yz(y+z) + y^3 + z^3$;
in this replacing $y + z$ by $x$, and transposing, we have

$$x^3 - 3yzx - (y^3+z^3) = 0. \qquad [2].$$

In order that [2] may be identical with [1] we must have,

$$3yz = -q, \ (1), \text{ and } y^3 + z^3 = -r, \ (2).$$

From (1) we have $y^3z^3 = -\dfrac{q^3}{27}$, and from (2), $y^3 + z^3 = -r$; thus the sum, $y^3 + z^3$, of two unknown quantities being given, and their product $y^3z^3$, we can determine the quantities by the quadratic equation $t^2 + rt - \dfrac{q^3}{27} = 0$, which is called the reducing equation. Solving this equation, we have

$$t_1 \text{ or } y^3 = -\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}},$$

$$t_2 \text{ or } z^3 = -\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}},$$

and, since $x = y + z$, we have the general formula for the roots,

$$\sqrt[3]{y^3} + \sqrt[3]{z^3} = \sqrt[3]{\frac{-r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}} + \sqrt[3]{\frac{-r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}} = x.$$

By Art. 113 the cube root of $y^3$ may be any of the three expressions,

$$y, \quad \tfrac{1}{2}(-1 + \sqrt{-3})\,y, \quad \tfrac{1}{2}(-1 - \sqrt{-3})\,y,$$

and the cube root of $z^3$ any of the three,

$$z, \quad \tfrac{1}{2}(-1 + \sqrt{-3})\,z, \quad \tfrac{1}{2}(-1 - \sqrt{-3})\,z,$$

where $y$ and $z$ are the arithmetical cube roots of $y^3$ and $z^3$; there would, therefore, be nine values of $x$, if we could take any one of the three values of $\sqrt[3]{y^3}$ with any one of the three values of $\sqrt[3]{z^3}$. But one of the conditions employed in obtaining the reducing equation was $yz = \frac{-q}{3}$, that is, the product of any pair of cube roots admissible in the solution must be a real quantity. It will be seen that the only pairs of the cube roots that satisfy this condition are,

$$x = y + z,$$
$$x = \tfrac{1}{2}(-1+\sqrt{-3})y + \tfrac{1}{2}(-1-\sqrt{-3})z = -\tfrac{1}{2}\{(y+z) + (y-z)\sqrt{-3}\},$$
$$x = \tfrac{1}{2}(-1-\sqrt{-3})y + \tfrac{1}{2}(-1+\sqrt{-3})z = -\tfrac{1}{2}\{(y+z) - (y-z)\sqrt{-3}\},$$

and these are the three roots of $x^3 + qx + r = 0$.

**119.** The occurrence of nine different values of $\sqrt[3]{y^3} + \sqrt[3]{z^3}$ is explained by the fact that in the process of solution we cubed the expression $yz = \frac{-q}{3}$. Now $y^3z^3$ may be the cube of any of the expressions $yz$, $\tfrac{1}{2}(-1 + \sqrt{-3})\,yz$, or $\tfrac{1}{2}(-1 - \sqrt{-3})\,yz$, so that we really obtain the roots of

three different cubic equations in the formula. The entrance of extraneous values into the solution of an equation that has been squared to free it from radicals is a familiar experience in the solution of quadratic and similar equations.

**120.** We have now to consider the arithmetic possibility of tho solution obtained.

(1). If $\frac{r^2}{4} + \frac{q^3}{27} > 0$, then $\sqrt{\frac{r^2}{4} + \frac{q^3}{27}}$ is possible, the values of $y$ and $z$ can be determined, and the equation has one real root, $y + z$, while the other roots $\frac{1}{2}\left\{(y + z) \pm (y - z)\sqrt{-3}\right\}$ are evidently imaginary.

Ex. 1. $$x^3 - 6x - 40 = 0.$$

$$\begin{aligned} x &= \sqrt[3]{20 + \sqrt{400 - 8}} + \sqrt[3]{20 - \sqrt{400 - 8}} \\ &= \sqrt[3]{39.799..} \qquad + \sqrt[3]{.2....} \\ &= 3.4142.... \qquad + .5848 = 3.999.... \end{aligned}$$

We infer that $x = 4$ is a root, and find upon trial that it is so. In this example the remaining roots are most readily found by depressing the proposed cubic to the quadratic $x^2 + 4x + 10 = 0$, the roots of which are $x = -2 \pm \sqrt{-6}$.

Ex. 2. $$x^3 + 12x + 4 = 0.$$

$$\begin{aligned} x &= \sqrt[3]{-2 + \sqrt{4 + 64}} + \sqrt[3]{-2 - \sqrt{4 + 64}} \\ &= \sqrt[3]{6.24621...} \qquad + \sqrt[3]{-10.24621...} \\ &= 1.84165.. - 2.17197... = -.33032... \end{aligned}$$

The other roots are most conveniently found by the formula; they are $x = -\frac{1}{2}(.33032... \pm 4.01363..\sqrt{-3})$.

(2). When $\frac{r^3}{4} + \frac{q^3}{27} = 0$, the formula reduces to

$$x = \sqrt[3]{\frac{-r}{2}} + \sqrt[3]{\frac{-r}{2}} = 2\sqrt[3]{\frac{-r}{2}}.$$

In this case, since $y$ and $z$ are equal, the formula for the remaining roots becomes $x = -\frac{1}{2}(y+z)$, that is, there are two equal roots.

Ex. 3. $$x^3 - 27x - 54 = 0.$$

$$x = \sqrt[3]{27 + \sqrt{729 - 729}} + \sqrt[3]{27 - \sqrt{729 - 729}},$$
$$= 2\sqrt[3]{27} = 6;$$

the remaining roots are $x = -3, \quad x = -3.$

(3). When $\frac{r^2}{4} + \frac{q^3}{27} < 0$, then $\sqrt{\frac{r^2}{4} + \frac{q^3}{27}}$ is impossible, and all the expressions for the roots are impossible. But we know that the equation, being of odd degree, must have at least one real root. Moreover, the expression $\sqrt[3]{y^3}$ must have a root of the form $m + n\sqrt{-1}$, and, as $z^3$ differs from $y^3$ only in the sign of the quadratic radical, $\sqrt[3]{z^3}$ must have a root $m - n\sqrt{-1}$. Substituting these in the expressions $y + z, \quad -\frac{1}{2}\{(y+z) \pm (y-z)\sqrt{-3}\}$, we obtain

$$2m, \text{ and } -m \pm n\sqrt{3};$$

the roots are therefore all real.

Ex. 4. $$x^3 - 57x - 56 = 0.$$

$$x = \sqrt[3]{28 + \sqrt{-6075}} + \sqrt[3]{28 - \sqrt{-6075}}$$
$$= \sqrt[3]{28 + 45\sqrt{-3}} + \sqrt[3]{28 - 45\sqrt{-3}}$$

We possess no arithmetical method of extracting these cube roots, though it will be found upon trial that

$$\sqrt[3]{28+45\sqrt{-3}} = 4+\sqrt{-3}; \text{ and } \sqrt[3]{28-45\sqrt{-3}} = 4-\sqrt{-3}.$$

Thus, in the case when the roots are all real and unequal, the formula is practically unavailable for purposes of arithmetical computation, since we are not able to perform the operations indicated. On this account this is sometimes called the Irreducible Case of Cardan's Formula.

**121.** That $y$ and $z$ are necessarily imaginary expressions, when all the roots are real and unequal, and what functions these expressions are of the roots, may be shown as follows:

Since the equation $x^3 + qx + r = 0$ has any one of its roots equal to the sum of the other two with contrary sign, the roots must be of the form $2a$, $-(a+\beta)$, $-(a-\beta)$, where $a$ is always real, while $\beta$, the semidifference of two of the roots, may be real, zero, or imaginary. Therefore we may put

$$x^3 + qx + r = x^3 - (3a^2+\beta^2)\,x - (2a^3-2a\beta^2) = 0\,;$$

from this identity we have

$$3a^2 + \beta^2 = -q \;\;(1), \qquad 2a^3 - 2a\beta^2 = -r \;\;(2),$$

from which to find $a$ and $\beta$ in terms of $q$ and $r$.

$$\text{Thus } \frac{r^2}{4} + \frac{q^3}{27} = (a^3-a\beta^2)^2 - (a^2+\tfrac{1}{3}\beta^2)^3$$
$$= -3a^4\beta^2 + \tfrac{2}{3}a^2\beta^4 - \tfrac{1}{27}\beta^6 = -3(a^2\beta - \tfrac{1}{9}\beta^3)^2\,;$$
$$\therefore\; \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} = \pm\sqrt{-3}\,(a^2\beta - \tfrac{1}{9}\beta^3).$$

Adding $-\dfrac{r}{2}$ and its equivalent $(a^3-a\beta^2)$ to the members of the above,

$$-\frac{r}{2} \pm \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} = (a^3 - a\beta^2) \pm \sqrt{-3}\,(a^2\beta - \tfrac{1}{9}\beta^2)$$
$$= a^3 \pm \sqrt{-3}\cdot a^2\beta - a\beta^2 \mp \frac{1}{\sqrt{-27}}\beta^3$$
$$= \left(a \pm \frac{\beta}{\sqrt{-3}}\right)^3,$$
$$\therefore\; \sqrt[3]{-\frac{r}{2} \pm \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}} = a \pm \frac{\beta}{\sqrt{-3}}\,;$$
$$\text{that is, } y = a + \frac{\beta}{\sqrt{-3}}, \quad z = a - \frac{\beta}{\sqrt{-3}}.$$

Thus we find that the expressions in Cardan's Formula are perfect cubes in regard to $a \pm \dfrac{\beta}{\sqrt{-3}}$. The difficulty is

that we possess no arithmetical method of obtaining these cube roots in a finite form even when $a$ and $\beta$ are commensurable. Had we any means of obtaining $a + \frac{\beta}{\sqrt{-3}}$ and $a - \frac{\beta}{\sqrt{-3}}$, their sum, $2a$, would give one root, and their difference multiplied by $\frac{1}{2}\sqrt{-3}$, would give $\beta$, enabling us to find the other roots $a \pm \beta$. We also see:

(1). When $\beta$ is real, the roots $2a$, $-(a \pm \beta)$, are all real; but $\frac{\beta}{\sqrt{-3}}$ is imaginary, and the formula necessarily assumes the so-called irreducible form.

(2). When $\beta$ is zero, the part under the quadratic radical vanishes, the formula becomes $x = \sqrt[3]{a^3} + \sqrt[3]{a^3}$; and the roots are, $2a$, $-a$, $-a$, *i. e.*, there are two equal roots.

(3). When $\beta$ is imaginary, $\frac{\beta}{\sqrt{-3}}$ is real, and there is one real root, $2a$, the others, $-(a \pm \beta)$, being imaginary. In this case, the quadratic radical in the formula being possible, we can find its square root, and then obtain approximate values for the cube roots whose sum $= x$.

**122.** We have seen that the criterion for the roots being all real is, that $\frac{r^2}{4} + \frac{q^3}{27}$, or $27r^2 + 4q^3$, must not be a negative quantity. With the notation employed in (121), we have $27r^2 + 4q^3 = -4(81a^4\beta^2 - 18a^2\beta^4 + \beta^6) = -4\beta^2(9a^2 - \beta^2)$ $= -4\beta^2(3a+\beta)^2(3a-\beta)^2$, which last expression is obviously the product of the squares of the differences of the roots. Hence, if the roots $2a$, $-(a \pm \beta)$ are all commensurable, the expression, $27r^2 + 4q^3$, must be a perfect square, which is the product of at least two squares different from unity. If there is but one commensurable root, it may be shown that the same expression must have at least one factor a perfect square different from unity.

**123.** Many vain attempts have been made to discover a process for obtaining in a finite form the cube roots of ex-

pressions of the form $m + n\sqrt{-1}$. But just as the extraction of the square root of such an expression involves the solution of a quadratic equation, so the extraction of the cube root of $m + n\sqrt{-1}$ requires the solution of a cubic equation, which will be found to be the original cubic that led to that expression.

**124.** The expression $(m + n\sqrt{-1})^{\frac{1}{3}}$ may, it is true, be expanded in a series by means of the Binomial Formula. We thus obtain an approximate value of $(m + n\sqrt{-1})^{\frac{1}{3}}$ of the form $P + Q\sqrt{-1}$; an approximate value of $(m - n\sqrt{-1})^{\frac{1}{3}}$ must therefore be $P - Q\sqrt{-1}$; thus we obtain $2P$ as an approximate value of $x$. This method, however, is of no practical use, as the calculations are laborious and the series are usually of slow convergence.

**125.** The roots of a cubic may also be obtained by means of a table of sines and cosines; we shall merely indicate the process.

By trigonometry we know that the equation

$$\cos^3\theta - \tfrac{3}{4}\cos\theta - \tfrac{1}{4}\cos 3\theta = 0, \qquad [1].$$

has for roots, $\cos\theta$, $\cos(120^\circ + \theta)$, $\cos(120^\circ - \theta)$.

Now if the proposed equation be $x^3 - qx - r = 0$, we can, by Art. 59, transform it into another the roots of which are those of the proposed each multiplied by $\sqrt{\dfrac{3}{4q}}$; thus,

$$y^3 - \tfrac{3}{4}y - r\sqrt{\frac{27}{64q^3}} = 0. \qquad [2].$$

If we assume [2] as identical with [1], we have

$$y = \cos\theta, \text{ and } r\sqrt{\frac{27}{64q^3}} = \tfrac{1}{4}\cos 3\theta, \text{ or } r\sqrt{\frac{27}{4q^3}} = \cos 3\theta.$$

As the cosine of no angle can be greater than unity, this last equation can hold true only when $27r^2$ is not greater than $4q^3$, *i. e.*, when all the roots are real. The roots of [2] are

therefore $\cos\theta$, $\cos(120^\circ+\theta)$, $\cos(120^\circ-\theta)$; and those of the proposed equation,

$$\sqrt{\frac{4q}{3}}\cdot\cos\theta,\quad \sqrt{\frac{4q}{3}}\cdot\cos(120^\circ\pm\theta).$$

To find these roots we determine from a table of cosines the angle whose cosine $= r\sqrt{\frac{27}{4q^3}}$ or $\sqrt{\frac{27r^2}{4q^3}}$. This angle will be $3\theta$, one-third of which is $\theta$: the cosine of $\theta$ multiplied by $\sqrt{\frac{4q}{3}}$ will be the greatest root; and similarly for the other roots.

EXERCISES.

Solve, by Cardan's Formula, the following equations :

1. $x^3 + 6x + 2 = 0.$
2. $x^3 - 12x - 20 = 0.$
3. $x^3 + 7x - 7 = 0.$
4. $x^3 + 11x + 8 = 0.$
5. $5x^3 - 13x - 52 = 0.$
6. $11x^3 - 56x + 105 = 0.$

**126.** In conclusion it may be observed that all attempts at an algebraical solution of the cubic, lead to a formula similar to, or identical with, that of Cardan. Among others, Tschirnhausen, Tischendorf and Lagrange, by very different paths, arrive at the same result. We shall indicate the following investigation, which the student may work out as an exercise. It may be easily shown that the equation $x^3 + 3x - (a^3 - a^{-3}) = 0$ has a root $a - \frac{1}{a}$. Any proposed cubic $x^3 + qx + r = 0$ may (59) be transformed to $y^3 + 3y + r\sqrt{\frac{27}{q^3}} = 0$. We have now to find $a$ from the equation $a^3 - a^{-3} = -r\sqrt{\frac{27}{q^3}}$, or $a^6 + \sqrt{\frac{27r^2}{q^3}}\cdot a^3 - 1 = 0$, and this, we find, leads to a formula for $x$ similar to that of Cardan.

In a subsequent chapter we shall take up the solution of cubics by Horner's Method, and it will be seen that the computation of a root of a cubic of the form $x^3 + qx + r = 0$ is a matter of no more difficulty than the arithmetical extraction of a cube root.

## BIQUADRATIC EQUATIONS.

**127.** Of the various algebraical solutions of the biquadratic that have been proposed, the first, historically, is that discovered by Ferrari, a pupil of Cardan's, soon after the publication of that solution of the cubic known as Cardan's.

FERRARI'S SOLUTION. — Let the proposed equation, deprived of its second term, be $x^4 + qx^2 + rx + s = 0$, then

$$x^4 = -qx^2 - rx - s.$$

Add to both sides of the equation $2kx^2 + k^2$, then

$$(x^2 + k)^2 = (2k - q)x^2 - rx + (k^2 - s).$$

We have now to determine $k$ so that the second member may be a complete square. In order that this may be the case, we must have $(2k-q)(k^2-s) = \frac{r^2}{4}$, or

$$k^3 - \tfrac{1}{2}qk^2 - sk + (\tfrac{1}{2}qs - \tfrac{1}{8}r^2) = 0.$$

From this, the so-called *reducing cubic*, having determined a value of $k$, by any of the methods given for the solution of cubics, the solution of the proposed biquadratic is reduced to that of the two quadratics,

$$\begin{aligned} x^2 + k &= \sqrt{2k-q} \cdot x - \sqrt{k^2 - s}, \\ x^2 + k &= -\sqrt{2k-q} \cdot x + \sqrt{k^2 - s}. \end{aligned}$$

The following solution is, however, generally more convenient.

**128.** DESCARTES' SOLUTION.— As before, let the proposed equation be

$$x^4 + qx^2 + rx + s = 0, \qquad [1]$$

and assume the first member to be the product of two quadratic factors $(x^2 + kx + f)$ and $(x^2 - kx + g)$, thus,

$$x^4 + (f + g - k^2)x^2 + (gk - fk)x + fg = 0. \qquad [2].$$

Equating coefficients of equal powers of $x$ in [1] and [2], we obtain the equations,

$$f + g - k^2 = q, \quad gk - fk = r, \quad fg = s. \qquad [3].$$

Having found $f$ and $g$ in terms of $k$ from the first two of these equations, and substituting in the third, we have

$$\left(k^2 + \frac{r}{k} + q\right)\left(k^2 - \frac{r}{k} + q\right) = 4s,$$

from which we obtain by reduction,

$$k^6 + 2qk^4 + (q^2 - 4s)k^2 - r^2 = 0,$$

or, putting $z$ for $k^2$, we have the reducing cubic,

$$z^3 + 2qz^2 + (q^2 - 4s)z - r^2 = 0.$$

When from this cubic a value of $z$ has been determined, its square root is $k$, and $k$ being known, $f$ and $g$ become known from the first two of the equations in [3], and we obtain the four roots of the biquadratic from the two quadratics, $x^2 + kx + f = 0$, $x^2 - kx + g = 0$.

Ex. $$x^4 - 24x^2 + 15x - 2 = 0.$$

The reducing cubic is,

$$z^3 - 48z^2 + 584z - 225 = 0.$$

From this we find one root, $z = 25$, $\therefore$ $k = 5$, $f = 2$, $g = -1$; and we can determine the four roots of the proposed from the equations, $x^2 + 5x + 2 = 0$, $x^2 - 5x - 1 = 0$.

**129.** We shall give one more solution, of some interest as proceeding by a method analogous to that pursued in Cardan's solution of the cubic; a method that has been applied to the solution of equations of higher degree than the fourth, and fails only in so far as that the reducing equations obtained are of higher degree than the original equation.

**130.** Euler's Solution.—Let the proposed equation be,

$$x^4 + qx^2 + rx + s = 0. \qquad [1].$$

Assume $\quad x = y + z + u, \quad$ then

$$x^2 = y^2 + z^2 + u^2 + 2(yz + yu + zu),$$

and $\quad x^2 - y^2 - z^2 - u^2 = 2(yz + yu + zu).$

Squaring both sides, we obtain

$$x^4 - 2(y^2+z^2+u^2)x^2 + (y^2+z^2+u^2)^2 = 4(yz+yu+zu)$$
$$= (y^2z^2 + y^2u^2 + z^2u^2) + 8yzu\,(y+z+u)$$

Replacing $y + z + u$ by $x$, and transposing, we have

$$x^4 - 2(y^2 + z^2 + u^2)\,x^2 - 8yzu \cdot x + (y^2 + z^2 + u^2)$$
$$- 4(y^2z^2 + y^2u^2 + z^2u^2) = 0. \qquad [2].$$

In order that [2] may be identical with [1], we must have

$$q = -2(y^2 + z^2 + u^2); \qquad r = -8yzu;$$
$$s = (y^2 + z^2 + u^2) - 4(y^2z^2 + y^2u^2 + z^2u^2),$$

or, $y^2+z^2+u^2 = -\dfrac{q}{2}$; $\quad y^2z^2+y^2u^2+z^2u^2 = \dfrac{1}{4}\left(\dfrac{q^2}{4} - s\right) = \dfrac{q^2-4s}{16}$;

$$y^2z^2u^2 = \frac{r^2}{64}.$$

The conditions show (45) that $y^2$, $z^2$, $u^2$, must be the roots of the cubic equation

$$t^3 + \frac{q}{2}t^2 + \frac{q^2 - 4s}{16} \cdot t - \frac{r^2}{64} = 0.$$

Denoting the roots of this equation by $t_1$, $t_2$, $t_3$, then

$$y = \pm\sqrt{t_1}; \qquad z = \pm\sqrt{t_2} \qquad u = \pm\sqrt{t_3}.$$

If we take $x = y + z + u = \pm\sqrt{t_1} \pm \sqrt{t_2} \pm \sqrt{t_3}$, we obtain *eight* different values for $x$. But some of these are inadmissible, since one of the conditions was $yzu = \dfrac{-r}{8}$, so that, supposing $r$ to be positive, the only admissible values of $x$ are,

$$-\sqrt{t_1}-\sqrt{t_2}-\sqrt{t_3}, \quad -\sqrt{t_1}+\sqrt{t_2}+\sqrt{t_3}, \quad \sqrt{t_1}-\sqrt{t_2}+\sqrt{t_3},$$
$$\sqrt{t_1}+\sqrt{t_2}-\sqrt{t_3},$$

namely those combinations the product of whose terms is negative.

If $r$ is negative, the admissible values of $x$ are,

$$\sqrt{t_1}+\sqrt{t_2}+\sqrt{t_3},\quad \sqrt{t_1}-\sqrt{t_2}-\sqrt{t_3},\quad -\sqrt{t_1}+\sqrt{t_2}-\sqrt{t_3},$$
$$-\sqrt{t_1}-\sqrt{t_2}+\sqrt{t_3}.$$

**131.** The occurrence of the *eight* values of $x$ in the solution has an explanation similar to that given for the appearance of *nine* values for $x$ in the solution by Cardan's Formula (119). In the course of the solution we employed $y^2z^2u^2 = \frac{r^2}{64}$; now $\frac{r^2}{64}$ may arise from $r$ either positive or negative, so that the solution contains the roots of the equation,

$$x^4 + qx^2 - rx + s = 0,$$

as well as those of the proposed equation.

It may be remarked that the reducing cubic found by Euler's Method will be found to coincide with that obtained by Descartes' Method, if we put $4t = k^2$.

EXERCISES.

1. $x^4 - 22x^2 - 21x + 30 = 0.$
2. $x^4 - 7x^2 + 26x - 40 = 0.$
3. $x^4 + 3x^2 + 2x + 3 = 0.$
4. $x^4 - 12x^3 + 40x^2 - 29x + 6 = 0.$
5. $5x^4 + 22x^3 + 17x^2 - 36x - 21 = 0.$

**132.** The preceding methods of solution depend upon special analytical artifices, and have no apparent bond of union with each other, or with the methods of solution employed for equations of lower degrees. By the method of investigation employed below it will appear that the algebraical solution of the quadratic, and all those given for the biquadratic above, are but particular cases of the solution of the general problem: *to reduce the solution of an equation of the $2n^{th}$ degree, to that of two equations of the $n^{th}$ degree.*

Let us suppose $f(x) = 0$, which is of the $2n^{th}$ degree, to be

the product of two factors, $\phi(x)$ and $\psi(x)$, each of the $n^{th}$ degree, then

$$f(x) = \phi(x)\,\psi(x) = \left\{\frac{\phi(x)+\psi(x)}{2}\right\}^2 - \left\{\frac{\phi(x)-\psi(x)}{2}\right\}^2 = 0. \qquad [1].$$

Every function of even degree may, therefore, be regarded as being the difference of two functions, which are complete squares. If we can, then, resolve $f(x)$ into the difference of two squares, we can reduce the solution of $f(x) = 0$ to that of two equations of one-half the degree of $f(x)$. For, transposing one of the squares in [1], and extracting the square root of both sides of the resulting equation, we have

$$\frac{\phi(x) + \psi(x)}{2} = \pm \frac{\phi(x) - \psi(x)}{2},$$

from which we obtain, according as we take the upper or lower sign with the second member,

$$\phi(x) = 0, \quad \text{or} \quad \psi(x) = 0,$$

two equations of one-half the degree of $f(x) = 0$.

**133.** Theoretically this may be done; but we can see in advance that, for equations of above the fourth degree, the reducing equations will be of higher degree than the proposed equation. For, from the $2n$ binomial factors of $f(x)$ may be formed $\dfrac{\lfloor 2n}{\lfloor n \lfloor n}$ factors $\phi(x)$ and $\psi(x)$, each of the $n^{th}$ degree (45), hence from $f(x)$ we may form the expressions $\phi(x) \pm \psi(x)$ in $\frac{1}{2}\dfrac{\lfloor 2n}{\lfloor n \lfloor n}$ different ways. It is evident, therefore, that the coefficients of the functions $\phi(x) \pm \psi(x)$ may have $\frac{1}{2}\dfrac{\lfloor 2n}{\lfloor n \lfloor n}$ different values, and will, consequently, have to be determined by equations of that degree. To find the degree of the reducing equations for equations of the second, fourth, and sixth degrees respectively, we substitute successively 1, 2, and 3, for $n$ in $\dfrac{\lfloor 2n}{2\lfloor n \lfloor n}$, and obtain 1, 3, and 10, as the numbers expressing the degrees of the respective reducing equations.

The quadratic and biquadratic may therefore be reduced by means of reducing equations of lower degree; but to solve a bicubic equation by means of two cubics, we should, except in particular cases (145), have first to form and solve a reducing equation of the tenth degree.

**134.** To illustrate the process of obtaining the reducing equation, it will be sufficient to suppose $f(x)$ of the sixth degree; then $\phi(x)$ and $\psi(x)$ will be each of the third degree.

$$\begin{aligned}
\text{Let}\quad \phi(x) &= x^3 + m_1 x^2 + m_2 x + m_3,\\
\psi(x) &= x^3 + n_1 x^2 + n_2 x + n_3,\\
\therefore\quad \left\{\frac{\phi(x)+\psi(x)}{2}\right\} &= x^3 + \frac{m_1+n_1}{2}x^2 + \frac{m_2+n_2}{2}x + \frac{m_3+n_3}{2}\\
\left\{\frac{\phi(x)-\psi(x)}{2}\right\} &= \pm\left\{\frac{m_1-n_1}{2}x^2 + \frac{m_2-n_2}{2}x + \frac{m_3-n_3}{2}\right\}
\end{aligned}$$

Changing $\frac{m_1+n_1}{2}$, &c., to $A$, $B$, $C$, and $\frac{m_1-n_1}{2}$, &c., to $a$, $b$, $c$, and equating, we have, since (131) $\phi(x) + \psi(x) = \pm\{\phi(x) - \psi(x)\}$,

$$x^3 + Ax^2 + Bx - C = \pm(ax^2 + bx + c).$$

Squaring both sides, and collecting terms, we have

$$\begin{aligned}
x^6 + 2Ax^5 + (A^2 - a^2 + 2B)x^4 &+ (2AB - 2ab + 2C)x^3\\
&+ (B^2 - b^2 + 2AC - 2ac)x^2 + (2BC - 2bc)x\\
&+ (C^2 - c^2) = 0, \qquad [1],
\end{aligned}$$

the equation of the sixth degree having its coefficients expressed as functions of the coefficients of $\frac{\phi(x) \pm \psi(x)}{2}$. From this we may deduce the corresponding equations of the fourth and second degree by omitting the letters that become zero; thus we obtain,

$$\begin{aligned}
x^4 + 2Ax^3 + (A^2 - a^2 + 2B)x^2 &+ (2AB - 2ab)x\\
&+ (B^2 - b^2) = 0, \qquad [2]\\
x^2 + 2Ax + (A^2 - a^2) = 0, &\qquad [3].
\end{aligned}$$

**135.** We obtain the reducing equations by equating the coefficients of [1]. [2], or [3] with $p$, $q$, $r$, $s$, &c., the coefficients of like powers of $x$ in the general equations. Except $A$ (which always $= \frac{1}{2}p$), the coefficients $B$, $C$, &c., of $\dfrac{\phi(x) + \psi(x)}{2}$ will have to be determined, as shown above, by equations of the $k^{th}$ degree, where $k = \dfrac{\lfloor 2n}{2\lfloor n \lfloor n}$, while $a$, $b$, $c$, the coefficients of $\pm \dfrac{\phi(x) - \psi(x)}{2}$, will, since they have the double sign, be involved in equations of the form $(y^2)^k + P(y^2)^{k-1} + Q(y^2)^{k-2} +$ &c. $= 0$, which may also be solved as equations of the $k^{th}$ degree.

**136.** It will be observed that, since in the equations [1], [2], [3], there are $2n - 1$ unknown quantities, $B$, $C$, .... $a$, $b$, $c$, .... there will be $2n - 1$ forms of the reducing equation, according as we eliminate for one or other of the unknowns; it will also be seen that one of these equations being formed, the others may be deduced from it by suitable substitutions. Thus there is but one form of the reducing equation for the quadratic, there are three distinct forms for the biquadratic, two of which have been given, and five for the bicubic.

**137.** We shall first find, by this method, the reducing equation for the quadratic.

Comparing the two forms of the quadratic,

$$x^2 + px + q = 0,$$
$$x^2 + 2Ax + (A^2 - a^2) = 0,$$

and equating corresponding coefficients, we have

$$A = \frac{p}{2}; \quad A^2 - a^2 = \frac{p^2}{4} - a^2 = q;$$

whence $a^2 = \pm \dfrac{p^2 - 4q}{4}$, which is the reducing equation, and we have for $\dfrac{\phi(x) - \psi(x)}{2} = \pm \dfrac{\phi(x) - \psi(x)}{2}$

$$x + \frac{p}{2} = \pm \frac{\sqrt{p^2 - 4q}}{2},$$

the usual form to which we reduce the quadratic, and equivalent to two equations of the first degree.

**138.** Again, comparing the equation of the fourth degree,

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

and $x^4+2Ax^3+(A^2-a^2+2B)x^2+(2AB-2ab)x+(B^2-b^2)=0,$

we have, $A=\frac{p}{2}$; $A^2-a^2+2B=q$; $2AB-2ab=r$; $B^2-b^2=s$.

From the second of these conditions we have

$$a^2 = \frac{p^2}{4} + 2B - q,$$

and $$a^2 = \left(\frac{pB-r}{2b}\right)^2 = \left(\frac{pB-r}{2\sqrt{B^2-s}}\right)^2$$

from the third and fourth; thence

$$\frac{p^2}{4} + 2B - q = \frac{p^2B^2 - 2prB + r^2}{4(B^2-s)}, \text{ whence}$$

$$8B^3 - 4qB^2 + (2pr - 8s)B - (p^2s - 4qs + r^2) = 0 \quad [1];$$

from which, if we put $p = 0$, we obtain

$$8B^3 - 4qB^2 - 8sB + (4qs - r^2) = 0,$$

the reducing cubic as found by Ferrari.

**139.** The reducing cubic involving $a$ may be found by elimination, as was the preceding one; but it is more readily found by substituting in [1] the value of $B$ in terms of $a$. The full equation thus obtained is,

$$a^6 + \left(2q - \frac{3p^2}{4}\right)a^4 + \left(\frac{3}{16}p^4 - p^2q + pr + q^2 - 4s\right)a^2 - \left(\frac{p^3 - 4pq + 8r}{8}\right)^2 = 0,$$

which becomes, putting $p = 0$,

$$a^6 + 2qa^4 + (q^2 - 4s)a^2 - r^2 = 0, \qquad [2]$$

the reducing cubic as found by Descartes and Euler.

**140.** The reducing cubic involving $b$ may either be found directly, or by substitution from the others; it is,

$$64b^6 + (32pr + 64s - 16q^2)b^4 + (4p^2r^2 - 32prs - 8p^3qs - 8qr^2)b^2 - (r^4 - 2p^2r^2s - p^4s^2) = 0,$$

which putting $p = 0$, and $4b^2 = z$, may be simplified to

$$z^3 + (4s - q^2)z^2 - 2qr^2z - r^4 = 0.$$

Having obtained any one root of any of the above reducing cubics, we can find the quantities $A$, $B$, $a$, $b$, from the relations $2A = p$, $2B = a^2 + q - A^2$; $2b = \dfrac{pB - r}{a}$, and then put the equation in the form,

$$\frac{\phi(x) + \psi(x)}{2} = \pm\frac{\phi(x) - \psi(x)}{2},$$

that is, $$x^2 + Ax + B = \pm(ax + b),$$

and solve by two quadratic equations.

**141.** We may examine what functions the roots of the reducing cubics are of those of the biquadratic.

Supposing the roots of the biquadratic to be $\alpha$, $\beta$, $\gamma$, $\delta$, and those of the reducing cubic of Ferrari, $B_1$, $B_2$, $B_3$, we have

$$B_1 = \frac{m_2 + n_2}{2} = \frac{\alpha\beta + \gamma\delta}{2},\quad B_2 = \frac{\alpha\gamma + \beta\delta}{2},\quad B_3 = \frac{\alpha\delta + \beta\gamma}{2};$$

that is, the roots of Ferrari's reducing cubic are the three possible functions of the form $\dfrac{\alpha\beta + \gamma\delta}{2}$. Similarly the roots of the reducing equation in $b$ are, since $b = \dfrac{m_2 - n_2}{2}$,

$$b_1 = \frac{\alpha\beta - \gamma\delta}{2},\quad b_2 = \frac{\alpha\gamma - \beta\delta}{2}.\quad b_3 = \frac{\alpha\delta - \beta\gamma}{2}.$$

Denoting the roots of Descartes' reducing cubic by $a_1^2$, $a_2^2$, $a_3^2$, then, since (133) $a = \dfrac{m_1 - n_1}{2}$, where $-m$ denotes the

sum of any two of the roots, and $-n$ the sum of the remaining two, we have,

$$\pm a_1 = \mp \tfrac{1}{2}(\alpha+\beta-\gamma-\delta), \quad \pm a_2 = \mp \tfrac{1}{2}(\alpha-\beta+\gamma-\delta),$$
$$\pm a_3 = \mp \tfrac{1}{2}(\alpha-\beta-\gamma+\delta), \text{ also } A = -\tfrac{1}{2}(\alpha+\beta+\gamma+\delta),$$

$$\therefore\ \tfrac{1}{2}(-A-a_1-a_2-a_3) = \alpha, \quad \tfrac{1}{2}(-A-a_1+a_2+a_3) = \beta,$$
$$\tfrac{1}{2}(-A+a_1-a_2+a_3) = \gamma, \quad \tfrac{1}{2}(-A+a_1+a_2-a_3) = \delta.$$

If we suppose $A = 0$, *i. e.*, $p = 0$, we obtain

$$\tfrac{1}{2}(-a_1-a_2-a_3) = \alpha, \text{ \&c.},$$

the same results we obtained in (130).

**142.** If the biquadratic has all its roots real, it is obvious, from consideration of what function of these the roots of the reducing equations are, that the latter are also all real.

If the roots of the biquadratic are all imaginary, they must be of the form $P \pm Q\sqrt{-1}$ and $R \pm S\sqrt{-1}$, and upon putting these for $\alpha$, $\beta$, $\gamma$, $\delta$, in the expressions,

$$B_1 = \frac{\alpha\beta+\gamma\delta}{2}, \text{ \&c.}, \qquad b_1^2 = \left(\frac{\alpha\beta-\gamma\delta}{2}\right)^2, \text{ \&c.},$$
$$a_1^2 = \left(\frac{\alpha+\beta-\gamma-\delta}{2}\right)^2, \text{ \&c.},$$

we find that all the roots of the reducing cubics are real in this case also.

If the biquadratic has two real, and two imaginary roots, we find, in the same way, that in this case each of the reducing cubics has two imaginary roots, or, possibly, two equal roots; that is, *the reducing cubic will not fall under the irreducible case when the biquadratic has two real and two imaginary roots.*

**143.** Upon consideration of the final terms of the different reducing cubics, we find that, if we have either $p^2s - 4qs + r^2 = 0$, $p^3 - 4pq + 8r = 0$, or $r^4 - 2p^2r^2s - p^4s^2 = 0$, the biquadratic $x^4 + px^3 + qx^2 + rx + s = 0$ may be solved at once by means of quadratic equations, for this is equivalent

to having either $B_1 = 0$, or $b_1 = 0$, or $a_1 = 0$, and we proceed as in (140).

Ex. 1. $\quad x^4 - 7x^4 - 60x^2 + 221x - 169 = 0.$

Here we find $p^2s - 4qs + r^2 = 0$. We proceed to reduce the equation to two quadratics as follows: $A = \frac{p}{2} = \frac{7}{2}$, $B_1 = 0$, $a_1^2 = A^2 + 2B - q = \frac{49}{4} + 0 + 60 = \frac{289}{4}$, $\therefore a_1 = \pm\frac{17}{2}$; $b_1^2 = B_1^2 - s = 169$, $\therefore b_1 = \pm 13$; hence we have,

$$x^2 - \frac{7}{2}x + 0 = \pm\left(\frac{17}{2}x + 13\right),$$

therefore we have

$$x^2 - 12x + 13 = 0, \text{ and } x^2 + 5x - 13 = 0.$$

Ex. 2. $\quad x^4 - 14x^3 + 63x^2 - 98x + 24 = 0.$

Here we find $p^3 - 4pq + 8r = 0$, therefore $a_1 = 0$. Proceeding as before, we find $A = -7$, $B = 7$, $b_1 = \pm 5$, and have the two equations,

$$x^2 - 7x + 7 = \pm 5.$$

Ex. 3. $\quad x^4 + 3x^3 - 4x^2 + 9x + 9 = 0.$

Here we find $r^4 - 2p^2r^2s - p^4s^2 = 0$, therefore $b_1 = 0$, $B = 3$, $a = \pm\frac{7}{2}$; hence we have $x^2 + \frac{3}{2}x + 3 = \pm\frac{7}{2}x$; therefore

$$x^2 - 2x + 3 = 0, \text{ and } x^2 + 5x + 3 = 0.$$

**144.** A cubic equation may be put in the form,

$$x^4 + px^3 + qx^2 + rx = 0,$$

that is, may be regarded as a biquadratic having one root $x = 0$. We may, accordingly, derive reducing equations from those given above for the biquadratic by simply leaving out the symbol $s$. We thus find the reducing equation for a cubic, found by this method, is itself a cubic. We also find that

when in a cubic equation, $x^3 + px^2 + qx + r = 0$, the relation $p^3 - 4pq + 8r = 0$ holds good, the equation may be reduced as in (143).

**145.** When we attempt to form the reducing equation for the bicubic, we obtain, as we might expect, an equation of the 10th degree. Thus, like all others that have been tried, this method fails to reduce equations of the fifth or higher degrees by means of reducing equations of lower degree than the proposed.

When special relations exist among the roots, reduction may be effected by equations much lower in degree. Thus in the equation, as found in (134),

$$x^6 + 2Ax^5 + (A^2 - a + 2B)x^4 + (2AB - 2ab + C)x^3$$
$$+ (B^2 - b^2 + 2AC - 2ac)x^2 + (2BC - 2bc)x + (C^2 - c^2) = 0,$$

if we put $a = 0$, *i. e.*, suppose that of the roots the sum of one-half the number is equal to the sum of the other half, we see by inspection of the coefficients above that the quantities $A$, $B$, $C$, $b$, and $c$, may then be found by the series of simple equations,

$$A = \tfrac{1}{2}p\,; \quad B = \tfrac{1}{8}(4q - p^2)\,; \quad C = \tfrac{1}{2}(r - pB)\,; \quad a = 0\,;$$
$$b = \sqrt{B^2 + pC - s}\,; \quad c = \sqrt{C^2 - v},$$

that is, *if in any equation of degree not greater than the sixth, the sum of one-half the number of roots be equal to the sum of the other half, the equation may be reduced by inspection.*

Ex. Let $x^6 - 10x^5 + 22x^4 + 11x^3 - 50x^2 - 152x - 96 = 0$ be an equation having the sum of one-half its roots $= 5$, then, by the simple equations above given, we find

$$A = -5,\ B = -\frac{3}{2},\ C = -2,\ a = 0,\ b = \pm\frac{17}{2},\ c = \pm 10,$$

and we can put the equation in the form,

$$x^3 - 5x^2 - \frac{3}{2}x - 2 = \pm\left(\frac{17}{2}x + 10\right),$$

whence

$$x^3 - 5x^2 - 10x - 12 = 0, \text{ and } x^3 - 5x^2 + 7x + 8 = 0.$$

If, in an equation of the fifth degree, the sum of three of the roots is equal to the sum of the remaining two, we can, in the same way, reduce the solution to that of a cubic and a quadratic equation.

Ex. 2. Let $x^5 + 14x^4 + 43x^3 - 48x^2 - 258x - 72 = 0$ be an equation having three of its roots equal to the other two; then, as in the preceding example, we find

$$A = 7,\ B = -3,\ C = -3,\ a = 0,\ b = \pm 15,\ c = \pm 3,$$

and we can put the equation under the form,

$$x^3 + 7x^2 - 3x - 3 = \pm (15x + 3),$$

whence we obtain the two equations,

$$x^3 + 7x^2 - 18x - 6 = 0, \text{ and } x^2 + 7x + 12 = 0.$$

## EXERCISES.

Depress the following equations by the methods exemplified in (143) and (145):

1. $x^4 + 2x^3 - 3x^2 - 92x - 529 = 0.$
2. $x^4 - 17x^3 + 30x^2 - 195x - 225 = 0.$
3. $x^4 - 8x^3 + 20x^2 - 16x - 21 = 0.$
4. $x^4 - 20x^3 + 117x^2 - 170x - 60 = 0.$
5. $x^4 + 7x^3 - 20x^2 + 35x + 25 = 0.$
6. $x^4 - 3x^3 + 0x^2 + 93x - 961 = 0.$
7. $x^5 + 24x^4 + 142x^3 - 29x^2 - 543x - 105 = 0.$
8. $x^6 - 6x^5 + 9x^4 - 16x^2 - 40x - 25 = 0.$

# CHAPTER VIII.

## STURM'S THEOREM.

**146.** In a preceding chapter were given several theorems regarding the limits between which the real roots of an equation must be found. Allusion was also made to methods proposed by Rolle and Waring for separating the roots. The objection to these methods was not only the excessive labor required in their practical application, but also the fact that they leave the number of real roots existing in an equation doubtful till we have actually ascertained the situation of each.

The method about to be explained is the only one hitherto proposed by which the number of real roots can be determined *a priori*. Comprising the solution of a problem that had engaged the attention of analysts for above two centuries, this method is distinguished as much by the simplicity of the process as by the exactitude of the results attained.

**147.** STURM'S FUNCTIONS. — DEF. — Let $X = 0$ be an equation of the $n^{th}$ degree, having no equal roots,* $X_1$ the first derived function of $X$, and $X_2, X_3, \ldots. X_n$, the series of remainders obtained in performing the operation of obtaining the greatest common measure of $X$ and $X_1$, care being taken to change the signs of each remainder before employing it as a divisor,† changing the sign of the last remainder also; the series of functions,

$$X, X_1, X_2, X_3, \ldots. X_n,$$

we shall refer to as Sturm's Functions.

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* If the proposed equation have equal roots, the fact will be discovered (92) by our arriving at a zero remainder in the course of the process. It will be seen (156) that, in such a case, the theorem holds good for the preceding remainders.

† In the ordinary operation for the G. C. M. it is merely a matter of convenience

**148.** STURM'S THEOREM.—Let $\alpha$ and $\beta$ be two numbers, such that $\alpha < \beta$, then the excess in the number of variations of sign in the series of functions,

$$X,\ X_1,\ X_2,\ X_3, \ldots . X_n,$$

when $\alpha$ is substituted for $x$, over the number of variations when $\beta$ is put for $x$, expresses the number of real roots of the equation $X = 0$ that lie between $\alpha$ and $\beta$.

Let $Q_1, Q_2, \ldots . Q_{n-1}$, denote the quotients obtained in the successive divisions; then, since the dividend is equal to the product of divisor and quotient, plus the remainder, or minus the remainder with its sign changed, we have the following relations existing among the functions:

$$\begin{array}{llll}
X & = Q_1\ X_1 & - X_2, & (1). \\
X_1 & = Q_2\ X_2 & - X_3, & (2). \\
X_2 & = Q_3\ X_3 & - X_4, & (3). \\
\cdot & \cdot \quad \cdot & \cdot \quad \cdot & \cdot \\
X_{r-1} & = Q_r\ X_r & - X_{r+1}, & (r). \\
\cdot & \cdot \quad \cdot & \cdot \quad \cdot & \cdot \\
X_{n-2} & = Q_{n-1} X_{n-1} & - X_n, & (n-1).
\end{array}$$

From these equations we deduce the following conclusions:

I. *Two consecutive functions cannot vanish simultaneously for any value of $x$.* For, if possible, suppose $X_1 = 0$ and $X_2 = 0$ at the same time. Then, by Eq. (2), we shall have $X_3 = 0$; and if $X_2 = 0$ and $X_3 = 0$, then, by Eq. (3), we have also $X_4 = 0$, and so on to the last equation, which will give $X_n = 0$; but this is impossible, since, by supposition, $X = 0$ has no equal roots, and therefore $X_n$ must be a numerical remainder, independent of $x$.

II. *When any function, except $X$, vanishes, the adjacent functions have contrary signs.* Suppose, for example, $X_r$ in Eq. $(r)$ vanishes, then we have $X_{r-1} = -X_{r+1}$.

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whether we do, or do not, change a remainder before employing it as a divisor; here this change is an essential part of the process. Subject to this condition we may multiply or divide a remainder by any convenient number as in the ordinary process.

III. *$X$ and $X_1$ have contrary signs immediately before $X$ vanishes for some value of $x$, as $x = c$, and immediately after, $X$ and $X_1$ have the same sign.*

Denote $X$ and $X_1$ by $f(x)$ and $f_1(x)$; put $c \pm h$ for $x$, then (7)

$$f(c \pm h) = f(c) \pm hf_1(c) + \text{\&c., and } f_1(c \pm h) = f_1(c) \pm \text{\&c.,}$$

and $h$ may be taken so small that the signs of $f(c \pm h)$ and $f_1(c \pm h)$ will be the same as those of the first term in their expansions that does not vanish. But $f(c) = 0$ by supposition, while, by I, $f_1(c)$ cannot vanish at the same time. Therefore, when $h$ is taken small enough, the signs of $f(c \pm h)$ and $f_1(c \pm h)$ are the same as those of $\pm hf_1(c)$, and $f_1(c)$ respectively, and these differ in sign when $h$ is negative, and *vice versa;* that is, $X$ and $X_1$ have contrary signs when $x = c - h$, and the same sign when $x = c + h$, $h$ being a quantity as small as we please.

Hence, as $x$ increases through $c$, a root of $X = 0$, one variation of sign is lost in the series of Sturm's Functions.

IV. *No variation of sign is either lost or gained when any function, except $X$, vanishes.*

For suppose $X_r$ to vanish for some value of $x$, then, by II, $X_{r-1}$ and $X_{r+1}$ have contrary signs, and with whichever of the two $X_r$ agreed in sign just before it vanished, it will agree with the other after it has passed through zero, and changed sign; thus no variation of sign is either lost or gained when $X_r$ passes through zero.

By I no two consecutive functions can vanish at the same time. If two or more, that are not consecutive, vanish for the same value of $x$, then, if $X$ be one of them, it follows, by III, that a variation of sign is lost as $x$ increases through that value; and, if $X$ is not one of them, by IV no variation is lost, or gained. Thus we see that Sturm's Functions never lose a variation of sign except when $x$ increases through a root of $X = 0$, and no variation is ever gained. Hence *the number of variations lost as $x$ increases from a value $\alpha$ to a greater value $\beta$ is equal to the number of real roots of the equation $X = 0$ that lie between $\alpha$ and $\beta$.*

**149.** SCHOLIUM I.—It may not be at once evident to the student how it is that $X$, which (III) is of the same sign as $X_1$ immediately after $x$ has increased through a root of $X = 0$, is again of the contrary sign before $x$ increases through the next root. This becomes clear upon considering that $X_1$, being the first derived function of $X$, must, by Art. 82, become zero, and therefore change sign for some value of $x$ between any two successive roots of $X = 0$. Thus $X_1$, which was of the same sign as $X$ just after $x$ increased through a root of $X = 0$, will be of the contrary sign before $x$ increases through the next root of $X = 0$. By IV, this vanishing of $X_1$, or of any of the functions of inferior degree, does not alter the *number* of variations of sign in the series of functions, but merely changes the *distribution* of the plus and minus signs in the series.

**150.** SCH. 2.—If for some value of $x$, any of the inferior functions, as $X_r$, becomes zero, we may omit that function when counting the variations; for the sign of the vanishing function must form a permanence with the sign of one or other of the adjacent functions.

**151.** COR. 1.—If in the series of Sturm's Functions we successively put $-\infty$ and $+\infty$ for $x$, the excess of the number of variations produced by the first substitution over the number produced by the second is equal to the whole number of real roots in the equation.

When $\pm\infty$ is put for $x$, the sign of a function is always the same as that of the highest power of $x$ in that function; the series of signs for $x = +\infty$ is therefore the same as the series of leading signs in the functions; the series of signs for $x = -\infty$ may be easily derived from the preceding, by changing the signs of the leading terms that contain odd powers of $x$.

**152.** COR. 2.—*In order that an equation of the $n^{th}$ degree may have all its roots real, it is necessary and sufficient that the functions be $n + 1$ in number, and all of the same sign in their leading terms.*

The functions will, in general, be $n + 1$ in number, since each remainder is usually only one degree lower than the preceding one. If, then, there be $n + 1$ functions, all having

the same leading sign, there will be $n$ variations of sign when $x = -\infty$, and no variation when $x = +\infty$; there are therefore $n$ real roots. If there be fewer than $n+1$ functions, then, even when the leading signs are all alike, the substitution of $-\infty$ for $x$ will give fewer than $n$ changes of sign, there are therefore fewer than $n$ real roots.

**153.** COR. 3.—*If the functions be $n + 1$ in number, but not all of the same leading sign, there are as many pairs of imaginary roots as there are variations in the leading signs.*

For suppose there are $m$ variations of sign in the leading terms, then for $x = +\infty$ there will be $m$ variations and $n - m$ permanences of sign. For $x = -\infty$ the foregoing variations become permanences, and the permanences, variations; that is, for $x = -\infty$ there are $n - m$ variations. The excess of the number of variations for $x = -\infty$ over the number for $x = +\infty$ is, therefore, $n - 2m$; thus there are $n - 2m$ real roots in the equation, and consequently $2m$ imaginary roots.

**154.** COR. 4.—*The number of positive real roots is equal to the excess of the number of variations in the signs of the final terms of the functions, over the number of variations of sign in the leading terms.*

For when $x = 0$, the functions reduce to their final terms and present as many variations of sign as these do, and for $x = +\infty$ the functions have the same signs as their leading terms; hence the number of variations in the signs of the final terms, diminished by the number of variations in the leading terms, represents the number of real roots between 0 and $+\infty$.

**155.** Thus, when once we have obtained Sturm's Functions, we can, by mere inspection of the signs, determine, not only how many of the roots are real and how many imaginary, but also how many of the former are positive or negative.

To ascertain the situation of each of the positive roots, we successively substitute in the functions the series of numbers 0, 1, 2, 3, .... &c., till we reach a number that makes the number of variations of sign in the series of functions equal to the number of variations in the leading terms. Each loss

of a variation in the signs of the series of functions indicates the passing over a root. If more than one root be found to lie between successive integers, we may proceed to substitute successive fractions till we separate the roots.

The situation of the negative roots is determined in a similar manner.

**156.** The theorem has been proved on the supposition that $X = 0$ has no equal roots. We shall now show how to proceed when in the operation for finding Sturm's Functions we obtain a zero remainder, showing that the equation has equal roots.

Let the equation $X = 0$ have equal roots, then $X$, $X_1$, $X_2$, and the functions that follow will have the last function $X_p$ as a common divisor. We can, by dividing $X$ by the common divisor, obtain the function which, equated to zero, will (94) contain all the roots without repetition. But this is not necessary, since the factor $X_p$, being common to all the preceding functions, its presence or absence will not affect the number of variations of sign for any given value of $x$. For, if $X_p$ is positive for that value of $x$, its presence as a factor will not affect the series of signs at all, while if $X_p$ is negative it will reverse all the signs. Hence, *when we discover that* $X = 0$ *has equal roots, by the fact of some remainder becoming zero, we can employ the preceding functions to determine the number and situation of the real roots that are unequal.*

The common factor $X_p$ which contains the remaining roots may be analyzed separately to ascertain how often each root is repeated.

**157.** If in the course of the operation for obtaining the functions $X_2$, $X_3$, &c., we come to one $X_r$, which we know cannot change its sign, we need not proceed to find the remaining functions. For in the demonstration of the theorem the necessary property of the last function is that it be incapable of becoming zero. As, by supposition, $X_r$ cannot vanish, the demonstration holds good for the functions from $X$ to $X_r$. The fact that a function cannot change sign is most readily observed in the case of a quadratic function, which will generally be the second from the final remainder.

As the labor of finding Sturm's Functions increases very rapidly with the degree of the equation, and especially with the last functions, it is of some importance to save, if possible, the labor of computing these functions.

Ex. 1. Let the equation proposed for analysis be,

$$\begin{aligned} X &= x^3 - 4x^2 - 11x + 43 = 0, \quad \text{then} \\ X_1 &= 3x^2 - 8x - 11, \\ X_2 &= 2x - 7, \quad \text{omitting a factor 49,} \\ X_2 &= 9. \end{aligned}$$

By Cor. 2, the roots are all real, and, Cor. 4, two are positive; we arrange the series of signs for different values of $x$ as in the following scheme:

| $x$ | $X$ | $X_1$ | $X_2$ | $X_3$ | *Variations* |
|---|---|---|---|---|---|
| 0 | + | — | — | + | 2 |
| 1 | + | — | — | + | 2 |
| 2 | + | — | — | + | 2 |
| 3 | + | — | — | + | 2 |
| 4 | — | + | + | + | 1 |
| 5 | + | + | + | + | 0 |

There are at first two variations of sign, and the number continues the same till, for $x = 4$, we have one variation only; a root therefore lies between 3 and 4. For $x = 5$ the remaining variation is lost; thus another root lies between 4 and 5.

It will be sufficient to substitute for $x$ in $X$ alone, when but one root remains of a certain sign, as a change of sign in the final term will show when the root is passed. Thus to find the situation of the negative root in this example, we need only substitute negative values for $x$ in $X$. We find this root to lie between $-3$ and $-4$.

Ex. 2. Let

$$\begin{aligned} X &= x^5 - 5x^3 + 10x^2 - 20x - 15 = 0, \quad \text{then} \\ X_1 &= x^4 - 3x^2 + 4x - 4, \quad \text{omitting a factor 5,} \\ X_2 &= 2x^3 - 6x^2 + 16x + 15, \\ X_3 &= 4x^2 + 55x + 53, \\ X_4 &= -3601x + 3431, \\ X_5 &= -. \end{aligned}$$

In this example the sign only of the final function is determined; for this function, being independent of $x$, does not change sign. As in this, so in other examples, it will be sufficient to perform as much of the numerical work as will enable us to determine the sign of the final remainder, which sign, when changed, will be the constant sign of the final function.

A glance at the leading signs informs us, Cor. 3, that the equation has two imaginary roots; by Cor. 4, there is but one positive, and therefore two negative roots. Arranging the series of signs as before, we have,

| $x$ | $X$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | *Variations* |
|---|---|---|---|---|---|---|---|
| $-4$ | $-$ | $+$ | $-$ | $-$ | $+$ | $-$ | 4 |
| $-3$ | $+$ | $+$ | $-$ | $-$ | $+$ | $-$ | 3 |
| $-2$ | $+$ | $-$ | $-$ | $-$ | $+$ | $-$ | 3 |
| $-1$ | $+$ | $-$ | $-$ | $+$ | $+$ | $-$ | 3 |
| 0 | $-$ | $-$ | $+$ | $+$ | $-$ | $-$ | 2 |
| 1 | $-$ | $-$ | $+$ | $+$ | $-$ | $-$ | 2 |
| 2 | $-$ | $+$ | $+$ | $+$ | $-$ | $-$ | 2 |
| 3 | $+$ | $+$ | $+$ | $+$ | $-$ | $-$ | 1 |

By noting where the losses of variations occur, we see that the roots are situated in the intervals, $(-4, -3)$, $(-1, 0)$, $(2, 3)$.

Ex. 3. Let $X = x^6 - 60x^4 + 10x^3 - 120x^2 + 6x - 1650 = 0$,

then
$$\begin{aligned}
X_1 &= x^5 - 40x^3 + 5x^2 - 40x + 1,\\
X_2 &= 4x^4 - x^3 + 16x^2 - x + 330,\\
X_3 &= 701x^3 - 68x^2 + 1959x + 314,\\
X_4 &= -67288x^2 + 553067x - 40501906,\\
X_5 &= 11031642677x + 917865361528,\\
X_6 &= +.
\end{aligned}$$

Here we see by inspection of the leading signs that there are four imaginary roots, and of the real roots one is positive, the other negative. We can, therefore, readily ascertain the situation of these roots from $X$ alone. The roots lie in the intervals $(-8, -7)$, $(7, 8)$.

**158.** This example, in which the coefficients are of no great magnitude, shows how laborious the method is when applied to equations of moderately high degree. From the examples, generally of the third and fourth degrees, given in most works in illustration of this method, the student is apt to receive a very inadequate idea of the amount of computation involved in its application. To fully illustrate this point, we give one more example, taken from Young's Equations, an example in which the step involving the computation of $X_5$ presents more than one hundred and fifty figures in a row.

Let the equation be $X = 0$, and

$$
\begin{aligned}
X &= x^6 + 378x^5 + 38189x^4 + 492368x^3 - 572554x^2 + 213720x - 26352, \\
X_1 &= 3x^5 + 945x^4 + 76378x^3 + 738552x^2 - 572554x + 106860, \\
X_2 &= 10673x^4 + 2036631x^3 + 23836942x^2 - 18302601x + 3405618, \\
X_3 &= 160625580788x^3 + 10061221870437x^2 - 7442477781884x + 1368314891916, \\
X_4 &= 92655788707137853054 1x^2 - 679427162023871692444x + 124552732633275966924, \\
X_5 &= 7506524618205128697 11x - 27523460219350600 0862, \\
X_6 &= 40413253842290604789305273169804831711095569696 9400.
\end{aligned}
$$

From which the roots are found to be all real, and to lie in the intervals

$$(0, 1);\ (0, 1);\ (0, 1);\ (-10, -20);\ (-160, -170);\ (-190, -200).$$

**159.** The labor of computing these functions is evidently very great, and that of substituting different values of $x$ in them, in order to ascertain the intervals in which the roots lie, is not much less, even when we content ourselves with determining between what integers they lie. In the above example the three roots between 0 and 1 are respectively .366600.., .366660.., and .366661.., all three concurring to four places of decimals, and two of them to five places, so that the task of separating them by means of fractional substitutions in Sturm's Functions would be even more laborious than that of obtaining the functions.*

Upon the whole we must reach the conclusion that this celebrated theorem,† though the most important and interesting addition to the Theory of Equations that has been made in the course of the past two centuries, and though offering an infallible means of determining, not only the number of real roots in an equation, but also the exact situation of each, is too laborious in application to be generally available for equations of above the fourth degree.

## EXERCISES.

Find the intervals in which lie the real roots of the following equations:

1. $x^3 - 6x^2 + 7x - 3 = 0.$
2. $x^3 - 8x^2 + 13x - 11 = 0.$
3. $5x^3 - 39x^2 + 24x - 25 = 0.$
4. $x^4 - 5x^3 + 3x^2 + 7x + 2 = 0.$
5. $x^4 - x^3 - x^2 + 6 = 0,$
6. $x^5 + x^3 - 2x^2 + 3x - 2 = 0.$

---

* It will be shown in the course of the next chapter, in connection with Horner's Method, how such roots may be readily separated without the aid of Sturm's Functions.

† The theorem was discovered by M. Sturm in 1829; received the mathematical prize from the Académie Royale des Sciences; was published in Memoires présentés . . . . . par des Savans Etrangers, 1835.

# CHAPTER IX.

## HORNER'S METHOD.*

**160.** In the preceding chapter a method has been given by which we can determine the situation of each real root in a proposed numerical equation. We have now to consider a method of remarkable simplicity and elegance, by which we can, after determining the first figure of a real root, ascertain its value exactly, if commensurable, or, if incommensurable, to any degree of approximation desired.

**161.** Suppose that in an equation $f(x) = 0$ a positive real root is $x = a + a' + a'' +$ &c., where $a$, $a'$, $a''$, &c., are its successive figures in descending order of the decimal scale, and that by a previous analysis, or by trial, we have ascertained the first figure $a$ of this root. We proceed to transform $f(x) = 0$ into another equation $f(a + x') = 0$, whose roots are those of $f(x) = 0$, each diminished by $a$. We now proceed to ascertain the next figure $a'$ of the root to which we are approximating, and then transform $f(a + x') = 0$ into another equation whose roots are each less by $a'$. Proceeding in this way, we determine in succession the figures $a$, $a'$, $a''$, &c., till we have either obtained the root exactly, or to as many figures as are necessary to the degree of approximation required.

The points in which the superiority of Horner's Method consists are: (1). The successive transformed equations are obtained from the preceding equation by the compact and easy process already given in Art. 69. (2). After the first figure of the root has been obtained, we are guided with more

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* This method, named after its author, Mr. Horner, of Bath, England, was published by him in the year 1819.

or less exactitude to the next figure by a process which shall presently be explained. (3). After a few figures of the root have been determined, we can very much contract the operation, which ultimately becomes a case of contracted division.

We shall now consider in order, I. The Method of Transformation; II. The Method of Trial Divisors; III. The Method of Contraction.

**162.** The Method of Transformation.—This has been already explained and illustrated in Art. 69; we apply it as follows:

Let $f(x) = 3x^3 - 1326x^2 + 6576x - 9177 = 0$, and suppose we have ascertained that a root lies between 500 and 400; we proceed to transform $f(x) = 0$ into $f(400 + x') = 0$, the roots of which are those of $f(x) = 0$, each diminished by 400.

$$\begin{array}{rrrrl}
3 & -1326 & +6576 & -9177 & \underline{|\,400} \\
 & \underline{1200} & \underline{-50400} & \underline{-17529600} & \\
 & -126 & -43824 & -17538777 & = f(400), \\
 & \underline{1200} & \underline{429600} & & \\
 & 1074 & 385776 = f_1(400), & & \\
 & \underline{1200} & & & \\
 & 2274 = \tfrac{1}{2}f_2(400). & & &
\end{array}$$

Hence the transformed equation, whose roots $= x - 400$, is

$$3x'^3 + 2274x'^2 + 385776x' - 17538777 = 0.$$

We could now proceed to find by trial the next figure of the root $(400 + x')$, and, when it is found, transform by the figure thus found; but we shall presently show how we may be guided to this figure without unnecessary trials.

**163.** It is important to remark that if the root $A$, to which we are approximating, is the only positive root having $a$ for its leading figure, then, in the transformed equation, $A - a$ will be the smallest positive root, and the only one whose leading figure is in that order of the decimal scale which is next

below the order of $a$. For each root whose leading figure was *less* than $a$ will be negative in the transformed equation, and all the roots whose leading figure was *greater* than $a$ will have their leading figures of higher orders than $a'$, the leading figure of $A-a$. Thus supposing the roots of an equation to be 7, 34, 49, 56.3, 78, 290, and we diminish each by 50, then we have, $-43$, $-16$, $-1$, 6.3, 28, 240, as the roots in the equation transformed by 50.

If we again diminish each of these roots by 6, we obtain $-49$, $-22$, $-7$, .3, 22, 234, as the roots in the second transformed equation. Thus the root towards which we are approximating becomes rapidly quite small as compared with the other roots. Similar remarks apply when there are two or more roots having the same leading figure.

**164.** Method of Trial Divisors.—These are similar to those employed in the evolution of arithmetical roots, and are usually deduced as follows:

*If $a+x'$ is a root of $f(x)=0$, and $x'$ is sufficiently small as compared with $a$, then $-\dfrac{f(a)}{f_1(a)}=x'$ approximatively.*

For, by Art. 69, when $a+x'$ is put for $x$ in $f(x)=0$, then

$$f(a+x')=C_n x'^n+\dots\tfrac{1}{2}f_2(a)x'^2+f_1(a)x+f(a)=0.$$

Now if $x'$ is small as compared with $a$, we may neglect all the terms involving higher powers of $x'$ than the first; thus we have $f_1(a)x'+f(a)=0$, or $-\dfrac{f(a)}{f_1(a)}=x'$ approximatively.

**165.** The proposition is generally stated as above. It would be better to state the second condition under the form, *if $x'$ is small compared with the other roots.* This will be more clearly seen from the following way of deducing the rule:

Let $\alpha$, $\beta$, $\gamma$, $\delta$, .... $\kappa$, and $x'$, be the roots of the transformed equation $f(a+x') = 0$, then, by Art. 45, we have,

$$-f\,(a) = \alpha\beta\gamma\delta\ldots\kappa x', \text{ and}$$
$$f_1(a) = \alpha\beta\gamma\delta\ldots\kappa + x'(\beta\gamma\delta\ldots\kappa + \alpha\gamma\delta\ldots\kappa + \&c.)$$
$$\therefore\ -\frac{f\,(a)}{f_1(a)} = \frac{\alpha\beta\gamma\delta\ldots\kappa x'}{\alpha\beta\gamma\delta\ldots\kappa + x'(\beta\gamma\delta\ldots\kappa + \alpha\gamma\delta\ldots\kappa + \&c.)} = x' \text{ approximatively,}$$

when $x'$ is small as compared with the other roots.

**166.** The degree of approximation will evidently depend upon how small $x'$ is as compared with the remaining roots, and will be in excess over, or defect from, the true value of $x'$, according as $\alpha\beta\gamma\delta\ldots\kappa$ and $x'(\beta\gamma\delta\ldots\kappa + \&c.)$ have contrary signs or the same sign.

Now in 163 we saw that, if $a + x'$ was the only root having a certain leading figure, then $x'$ is, even in the first transformed equation, a decimal as compared with the other positive roots. Of the negative roots some may be numerically less than $x'$ in the first transformation (see 163), but at the same time these negative roots tend to diminish the expression $x'(\beta\gamma\delta\ldots\kappa + \alpha\gamma\delta\ldots\kappa + \&c.)$ by making the products vary in sign.

Thus upon the whole it will be found that, even in the first transformed equation, if we divide minus the final term by the preceding coefficient, the figure thus obtained will not be far from correct; in many instances will be exactly so. In the second transformed equation the trial divisor will give the next figure of the root correctly, with comparatively rare exceptions.

If the figure obtained is too great, we shall become aware of the fact before completing the first row of work in the next transformation, by seeing that the final sign will be changed, thus showing that the root has been passed over. We then try the next lower figure, and so on until we find one that does not cause a change in the final sign. If the figure obtained is too small, we shall not be warned of the fact in the same way, though we can generally infer, from the result produced

on the final term, whether the figure would require to be increased. In case of doubt, it would be prudent to try in a bye-operation, whether the next higher figure would change the final sign, which it ought to do if the figure obtained by the trial divisor is correct. Finally, if at any part of the operation we transform by a figure too small, we shall be informed of the fact by the next trial divisor giving a figure of too high order to be correct.

**167.** Resuming consideration of the transformed equation of 162,

$$3x'^3 + 2274x'^2 + 385776x' - 17538777 = 0,$$

we find that $-\dfrac{f(400)}{f_1(400)}$, or $\dfrac{1753\ldots}{385\ldots.} = 40 +$; we therefore transform by 40 as follows:

$$\begin{array}{rrrrl} 3 & +2274 & +385776 & -17538777 & \underline{|40} \\ & \underline{120} & \underline{95760} & & \\ & 2394 & 481536 & & \end{array}$$

We need proceed no further with this transformation, as it is obvious that, if completed, the final sign will be changed. We accordingly transform by 30, thus:

$$\begin{array}{rrrrl} 3 & +2274 & +385776 & -17538777 & \underline{|30} \\ & \underline{90} & \underline{70920} & \underline{13700880} & \\ & 2364 & 456696 & -3837897 & \\ & \underline{90} & \underline{73620} & & \\ & 2454 & 530316 & & \\ & \underline{90} & & & \\ & 2544 & & & \end{array}$$

and have the transformed equation

$$3x''^3 + 2544x''^2 + 530316x'' - 3837897 = 0.$$

As $-\dfrac{f(430)}{f_1(430)}$, or $\dfrac{3837\ldots}{530\ldots.} = 7 +$, we transform by 7, thus:

| | | | | |
|---|---|---|---|---|
| 3 | +2544 | +530316 | −3837897 | 7 |
| | 21 | 17955 | 3837897 | |
| | 2565 | 548271 | | |
| | 21 | 18102 | | |
| | 2586 | 566373 | | |
| | 21 | | | |
| | 2607 | | | |

Hence the equation has a root 437, since $f(437) = 0$. The remaining roots may be determined from the depressed equation $3x'''^2 + 2607x''' + 566373 = 0$, the roots of which, when increased by 437, are roots of $f(x) = 0$.

For the sake of perspicuity we have exhibited each transformation separately in the preceding example; in practice we proceed as follows:

| | | | | |
|---|---|---|---|---|
| 3 | −1326 | + 6576 | −9177 | 400+30+7 |
| | 1200 | −50400 | $-17529600_1$ | |
| | −126 | −43824 | −17538777 | |
| | 1200 | $429600_1$ | $13700880_2$ | |
| | 1074 | 385776 | −3837897 | |
| | $1200_1$ | 70920 | 3837897 | |
| | 2274 | 456696 | | |
| | 90 | $73620_2$ | | |
| | 2364 | 530316 | | |
| | 90 | 17955 | | |
| | 2454 | 548271 | | |
| | $90_2$ | $18102_3$ | | |
| | 2544 | 566373 | | |
| | 21 | | | |
| | 2565 | | | |
| | 21 | | | |
| | 2586 | | | |
| | $21_3$ | | | |
| | 2607 | | | |

indicating the corresponding terms of each transformation by a small index figure, or other mark.

**168.** Ex. 2. $f(x) = x^4 - 5x^3 + 2x^2 - 13x + 55 = 0$ has a root lying between 2 and 3; we proceed as follows:

| | | | | | |
|---|---|---|---|---|---|
| 1 | $-5$ | $+2$ | $-13$ | $+55$ | 2.3 |
| | $2$ | $-6$ | $-8$ | $-42_1$ | |
| | $-3$ | $-4$ | $-21$ | $13$ | |
| | $2$ | $-2$ | $-12_1$ | $-10.1709_2$ | |
| | $-1$ | $-6$ | $-33$ | $2.8291$ | |
| | $2$ | $2_1$ | $-.903$ | | |
| | $1$ | $-4$ | $-33.903$ | | |
| | $2_1$ | $.99$ | $-.579_2$ | | |
| | $3$ | $-3.01$ | $-34.482$ | | |
| | $.3$ | $1.08$ | | | |
| | $3.3$ | $-1.93$ | | | |
| | $.3$ | $1.17_2$ | | | |
| | $3.6$ | $-.76$ | | | |
| | $.3$ | | | | |
| | $3.9$ | | | | |
| | $.3_2$ | | | | |
| | $4.2$ | | | | |

In the first transformation we found $\dfrac{-f(2)}{f_1(2)}$, or $\dfrac{13}{33} = .3+$, and transformed again by .3, as above. As we find $\dfrac{-f(2.3)}{f_1(2.3)}$, or $\dfrac{2.8291}{34.482} = .08+$, we could proceed to transform again by .08, and so on for the remaining figures of the root.

**169.** In practice it is found convenient, however, to dispense with the decimal points. This we effect as follows: *When, in an equation of the $n^{th}$ degree, we have diminished the root, to which we are approximating, till the next figure would be a decimal, we add n ciphers to the right hand working column when completed, $(n-1)$ ciphers to the next column when completed, and so on.* This (59) is equivalent to multiplying the roots each by 10, and by repeating this multiplication at every transformation we are enabled to dispense with the troublesome decimal points in the working columns, the ciphers also serving to mark where a column was completed.

Beginning the example again, we proceed thus:

| | | | | | |
|---|---|---|---|---|---|
| 1 | −5 | +2 | −13 | +55 | 2.381 |
| | 2 | −6 | − 8 | −42 | |
| | −3 | −4 | −21 | 130000 | |
| | 2 | −2 | −12 | −101709 | |
| | −1 | −6 | −33000 | 282910000 | |
| | 2 | 2 | −903 | −276123264 | |
| | 1 | −400 | −33903 | 67867360000 | |
| | 2 | 99 | −579 | −34520621079 | |
| | 30 | −301 | −34482000 | 333467389210000 | |
| | 3 | 108 | −33408 | | |
| | 33 | −193 | −34515408 | | |
| | 3 | 117 | −5504 | | |
| | 36 | −7600 | −34520912000 | | |
| | 3 | 3424 | 290921 | | |
| | 39 | −4176 | −34520621079 | | |
| | 3 | 3488 | 295443 | | |
| | 420 | −688 | −34520325636000 | | |
| | 8 | 3552 | | | |
| | 428 | 286400 | | | |
| | 8 | 4521 | | | |
| | 436 | 290921 | | | |
| | 8 | 4522 | | | |
| | 444 | 295443 | | | |
| | 8 | 4523 | | | |
| | 4520 | 29996600 | | | |
| | 1 | | | | |
| | 4521 | | | | |
| | 1 | | | | |
| | 4522 | | | | |
| | 1 | | | | |
| | 4523 | | | | |
| | 1 | | | | |
| | 45240 | | | | |

In every instance in this example we have found the trial divisor give the correct figure for the next transformation, and

we could go on in the same way to find as many more figures of the root as may be desired. But when we examine the second working column, that which we employ as trial divisor, we observe that the addends from the column to the left have less and less influence upon the leading figures of this column. It is evident, indeed, that if we continue to perform the operation without contraction, we shall soon have a large number of superfluous figures in the second and right-hand columns, which figures might be dispensed with without affecting the correctness of the result within certain limits. We therefore contract the work in the manner about to be explained.

**170.** METHOD OF CONTRACTION.—*When we have obtained a root to two places of decimals by the uncontracted process, we can contract the operation by ceasing to add ciphers to the right-hand column when completed, by striking off one figure from the second column when completed, two from the third, three from the fourth, and so on.*

This, it will readily be seen, is equivalent to first supposing the coefficients formed as in the preceding article, and then dividing each by $10^n$ ($n$ being the degree of the equation). Taking, for example, the coefficients of the third transformation above,

$$1 \quad +4520 \quad +286400 \quad -34520912000 \quad +67867360000,$$

and dividing each by 10,000, we have

$$.0001 \quad +.452 \quad +28.64 \quad -3452091.2 \quad +6786736.$$

With these contracted coefficients we proceed as before:

```
|452   +28|64   −3452091|2     +6786736  |196012
            4         29       −3452062
       ------   -----------    --------
        |29     −3452062        3334674
                      29       −3106800
                -----------    --------
                −345203|3        227874
                       3        −207118
                -----------    --------
                −345200           20756
                       3         −20712
                -----------    --------
                −3|4|5|1|9|7         44
                                    −35
                                    ---
                                      9
```

In the first step of this contracted operation, the trial divisor $\left(\frac{678\ldots}{345\ldots}\right)$ indicates 1 as the next figure of the root. Though we have struck off all the figures of the fourth working column, yet (as in contracted multiplication) we allow for its influence upon the third column, which thus becomes 29. We complete the transformation by 1, and strike off figures as before. The trial divisor now indicates 9 as the next figure; we take no further notice of the fourth column, but, though we have struck off the remaining two figures of the third column, we carry 3 from it to the second column, since $.29 \times 9 = 3$ to the nearest decimal. We then complete the transformation, and strike off another figure from the second column. The rest of the work is merely a contracted division. The approximated root thus obtained is $x = 2.381966012\ldots$, and is correct to the last figure, which would be 1 if obtained by the uncontracted process.

**171.** Another root of the same equation lies between 4 and 5; we proceed as on the following page to compute this root, contracting from the third transformation.

Observe here that the sign of the final term is changed in the first transformation. This arises from the fact that a root has been passed, that which we computed in 170. We observe also that the trial divisor suggests a number far above the true value of $x'$. This arises from the fact that $x'$ is not, in this transformation, much smaller, numerically, than the next root.

In the remaining transformations the trial divisor invariably suggests the true figure. The root thus obtained,

$$x = 4.61803399\ldots,$$

is correct to the nearest decimal.

**172.** To approximate to the negative roots of an equation, we change the signs of alternate terms, thus (61) changing negative to positive roots, and proceed as in the above examples. The roots thus obtained will, taken with the negative sign, be the negative roots of the equation.

<table>
<tr><td>1</td><td>−5</td><td>+2</td><td>−13</td><td>+55</td><td>4.618033996..</td></tr>
<tr><td></td><td>4</td><td>−4</td><td>− 8</td><td>−84</td><td></td></tr>
<tr><td></td><td>−1</td><td>−2</td><td>−21</td><td>·−290000</td><td></td></tr>
<tr><td></td><td>4</td><td>12</td><td>40</td><td>275856</td><td></td></tr>
<tr><td></td><td>3</td><td>10</td><td>19000</td><td>−141440000</td><td></td></tr>
<tr><td></td><td>4</td><td>28</td><td>26976</td><td>77944941</td><td></td></tr>
<tr><td></td><td>7</td><td>3800</td><td>45976</td><td>−63495059</td><td></td></tr>
<tr><td></td><td>4</td><td>696</td><td>31368</td><td>63224739</td><td></td></tr>
<tr><td></td><td>110</td><td>4496</td><td>77344000</td><td>−270320</td><td></td></tr>
<tr><td></td><td>6</td><td>732</td><td>600941</td><td>238542</td><td></td></tr>
<tr><td></td><td>116</td><td>5228</td><td>77944941</td><td>−31778</td><td></td></tr>
<tr><td></td><td>6</td><td>768</td><td>602283</td><td>23856</td><td></td></tr>
<tr><td></td><td>122</td><td>599600</td><td>7854722|4</td><td>−7922</td><td></td></tr>
<tr><td></td><td>6</td><td>1341</td><td>48368</td><td>7156</td><td></td></tr>
<tr><td></td><td>128</td><td>600941</td><td>7903090</td><td>−766</td><td></td></tr>
<tr><td></td><td>6</td><td>1342</td><td>48450</td><td>716</td><td></td></tr>
<tr><td></td><td>1340</td><td>602283</td><td>79515|4|0</td><td>−50</td><td></td></tr>
<tr><td></td><td>1</td><td>1343</td><td>2</td><td>48</td><td></td></tr>
<tr><td></td><td>1341</td><td>6036|26</td><td>79517</td><td>−2</td><td></td></tr>
<tr><td></td><td>1</td><td>10</td><td>2</td><td></td><td></td></tr>
<tr><td></td><td>1342</td><td>6046</td><td>7|9|5|1|9</td><td></td><td></td></tr>
<tr><td></td><td>1</td><td>10</td><td></td><td></td><td></td></tr>
<tr><td></td><td>1343</td><td>6056</td><td></td><td></td><td></td></tr>
<tr><td></td><td>1</td><td>10</td><td></td><td></td><td></td></tr>
<tr><td></td><td>1|344</td><td>60|66</td><td></td><td></td><td></td></tr>
</table>

**173.** By this compact operation we are thus enabled, when once the leading figure has been obtained, to approximate to any real root of a proposed equation with just the same amount of numerical labor that would be required for the extraction of an arithmetical root of an order corresponding to the degree of the equation. The extraction of the $n^{th}$ root of a given number $N$ is, in fact, nothing else than the solution of an equation of the form $x^n - N = 0$, and may be performed by the same process, and with the same contractions, employed in the above examples.

**174.** We have thus far supposed the root to which we wish to approximate to be the only one beginning with a certain leading figure. We can, it is true, by means of Sturm's Method, always separate real roots, however closely they may coincide. But, as shown in 159, the labor involved in doing so is excessive. It is now proposed to be shown that, guided by an extension of the rule above given in regard to trial divisors, we can safely proceed in the approximation by Horner's Method, even when there are in the given interval several equal or nearly equal roots.

**175.** It has been shown in Art. 164, *et seq.*, that in the transformed equation

$$f(a+x') = C_n x'^n + \dots \tfrac{1}{2} f_2(a)x'^2 + f_1(a)x' + f(a) = 0,$$

when $x'$ is small compared with $a$, the part of the root already found, we may neglect all but the two last terms, $f_1(a) + f(a) = 0$, from which simple equation we may obtain one or more figures of $x'$. Now if there are *two* nearly equal roots in the interval we are examining in $f(a+x') = 0$, then, since (82), the first derived equation* $f_1(a+x') = 0$ has a root lying between them, we should (164) obtain an approximation to this root from the last two terms of

$$f_1(a+x') = nC_n x^{n-1} + \dots \tfrac{1}{2} f_3(a)x'^2 + f_2(a)x' + f_1(a) = 0,$$

that is, $-f_1(a) \div f_2(a) = x'$ approximatively when there are two roots having figures in common.

Generally, if $f(a + x') = 0$ has $r$ equal or nearly equal roots, of which $a$ is the leading part, then (82) the $(r-1)^{th}$ derived equation has a root lying between the two least of these roots; that is,

$$f_{r-1}(a+x') = \dots\dots f_r(a)x' + f_{r-1}(a) = 0$$

has a root a little greater than the least of the $r$ roots of $f(a+x') = 0$, and, by Art. 164, an approximation to this root is given by the expression $-f_{r-1}(a) \div f_r(a)$. Therefore, writing the transformed equation as

$$C_n x'^n + \dots C'_r x'^r + C'_{r-1} x'^{r-1} + \dots C'_3 x'^3 + C'_2 x'^2 + C'_1 x' + C'_0 = 0.$$

* By the $r^{th}$ derived equation is meant the $r^{th}$ derived function equated to zero.

and remembering that $C'_r = \frac{1}{\underline{|r}} f_r(a)$, we have

$$-\frac{C'_0}{C'_1},\quad -\frac{C'_1}{2C'_2},\quad -\frac{C'_2}{3C'_3},\ldots\ldots -\frac{C'_{r-1}}{rC'_r}, \qquad \text{A}$$

as the respective expressions for suggesting the leading figure of the next transformation, according as there are one, two, three or $r$ roots in the interval under consideration.

We deduce, as follows, another set of expressions, which serve as a check upon those preceding.

Let $f(a+x') = \phi(x')\,\psi(x')$, where $\psi(x')$ is the product of $r$ equal factors $(x'-\kappa)$ and $\phi(x')$ the product of the $n-r$ other factors, then putting

$$\begin{aligned}
\phi(x') &= C_n x'^{n-r} + \ldots\ldots\ldots D_1 x' + D_0,\\
\psi(x') &= x'^r - r\kappa x'^{r-1}\ldots \pm r\kappa^{r-1}x' \mp \kappa^r,\\
\therefore f(a+x') &= C_n x'^n + \ldots\ldots\ldots \pm(rD_0\kappa^{r-1} - D_1\kappa^r)x' \mp D_0\kappa^r = 0,\\
&= C_n x'^n + \ldots\ldots\ldots + C'_1 x' + C'_0 = 0;
\end{aligned}$$

$\therefore\ -\frac{C'_0}{C'_1} = \frac{D_0\kappa}{rD_0 - \kappa D_1} = \frac{\kappa}{r}$, or $\frac{x'}{r}$, nearly when $\kappa$ is small as compared with the other roots.

If the factors in $\psi(x')$ are not absolutely, but yet nearly, equal, the above conclusion will still hold; thus the expressions

$$-\frac{2C'_0}{C'_1},\quad -\frac{3C'_0}{C'_1},\ldots\ldots -\frac{rC'_0}{C'_1}, \qquad \text{B}$$

ought to suggest the same value of $x'$ as the corresponding expressions in A, according as there are two, three, .... or $r$ equal, or nearly, roots in the interval under examination.

**176.** When, therefore, we have ascertained that there are several roots beginning with a common leading figure, we proceed as follows:

(1). Transforming in the usual way, we diminish the roots by this common figure.

(2). By means of the proper expression in A we determine the next figure. If the expression (A) at this stage suggests the exact figure, that from (B) will usually agree with it.

(3). We continue, at each transformation, to ascertain, in the same way, the common leading figures in succession, till either a discrepancy in the figures suggested by the expressions (A) and (B) warns us that the roots no longer coincide, or a change in the final sign shows that we have separated a root, which will be the least of those to which we were approximating.

(4). When we have thus separated a root, or group of roots, as the two groups do not differ much from each other in value, the divisors will not again be effective till we have ascertained the correct leading figure of each group and performed a transformation by that figure. We may then proceed separately with the approximation of each group, guided by the expressions in (A) and (B) appropriate to the number of roots in the group, till we separate these roots also, and carry on the approximation to each single root by means of the ordinary trial divisor as far as desired.

Ex. 1. The equation

$$x^4 - 437x^2 + 3660 + 2826 = 0$$

has two roots between 12 and 13; we proceed to develop them as shown on the next page.

In the first transformation we find $-\frac{2C'_0}{C'_1}$, or $\frac{4680}{636} = 7+$, and $-\frac{-C'_1}{2C_2}$, or $\frac{6360}{794} = 8+$; we transform by 7, and find that figure correct. In the next transformation we find both $\frac{341 \times 2}{826}$ and $\frac{826}{500 \times 2}$ suggest $0.8$; we accordingly, considering 0 as a figure of the root, add other ciphers to the working columns, and transform by 8. In the next transformation we begin to contract; both $\frac{293 \times 2}{246}$ and $\frac{246}{50 \times 2}$ indicate 2 as the next figure, In the next transformation, after striking off the contraction figures, we find $\frac{171 \times 2}{456}$ indicates 0, and $\frac{456}{50 \times 2}$ indicates 4. The roots therefore coincide no further; we find that 1 produces a change of sign in the final term, showing

<table>
<tr><td>1</td><td>+0</td><td>—467</td><td>+3660</td><td>+2826</td><td>12.708203932</td></tr>
<tr><td></td><td>12</td><td>144</td><td>—3876</td><td>—2592</td><td></td></tr>
<tr><td></td><td>12</td><td>—323</td><td>—216</td><td>2340000</td><td></td></tr>
<tr><td></td><td>12</td><td>288</td><td>—420</td><td>—2339659</td><td></td></tr>
<tr><td></td><td>24</td><td>—35</td><td>—636000</td><td>34100000000</td><td></td></tr>
<tr><td></td><td>12</td><td>432</td><td>301763</td><td>—34070626304</td><td></td></tr>
<tr><td></td><td>36</td><td>39700</td><td>—334237</td><td>29373696</td><td></td></tr>
<tr><td></td><td>12</td><td>3409</td><td>325969</td><td>—29202080</td><td></td></tr>
<tr><td></td><td>480</td><td>43109</td><td>—8268000000</td><td>171616</td><td></td></tr>
<tr><td></td><td>7</td><td>3458</td><td>4009171712</td><td>—132325</td><td></td></tr>
<tr><td></td><td>487</td><td>46567</td><td>—4258828288</td><td>39291</td><td></td></tr>
<tr><td></td><td>7</td><td>3507</td><td>4012423936</td><td>—37937</td><td></td></tr>
<tr><td></td><td>494</td><td>500740000</td><td>—24640435|2</td><td>1354</td><td></td></tr>
<tr><td></td><td>7</td><td>406464</td><td>10039395</td><td>—1251</td><td></td></tr>
<tr><td></td><td>501</td><td>501146464</td><td>—14601040</td><td>103</td><td></td></tr>
<tr><td></td><td>7</td><td>406528</td><td>10039598</td><td>—84</td><td></td></tr>
<tr><td></td><td>50800</td><td>501552992</td><td>—45614|4|2</td><td>19</td><td></td></tr>
<tr><td></td><td>8</td><td>406592</td><td>1505|9</td><td></td><td></td></tr>
<tr><td></td><td>50808</td><td>5019595|84</td><td>—44108|5</td><td></td><td></td></tr>
<tr><td></td><td>8</td><td>101|6</td><td>1506</td><td></td><td></td></tr>
<tr><td></td><td>50816</td><td>5019697|4</td><td>—4260|2</td><td></td><td></td></tr>
<tr><td></td><td>8</td><td>101|6</td><td>45|</td><td></td><td></td></tr>
<tr><td></td><td>50824</td><td>5019799</td><td>—4215</td><td></td><td></td></tr>
<tr><td></td><td>8</td><td>102</td><td>45</td><td></td><td></td></tr>
<tr><td></td><td>50|832</td><td>5|01|98|01</td><td>—4|1|7|0</td><td></td><td></td></tr>
</table>

that the figure of the lesser root is 0. We carry the contraction a step further, and by the ordinary trial expression $\frac{171}{45}$ have 3 suggested as the next figure, which proves to be correct. Carrying on the approximation for a few steps further, the correct figures being indicated in succession by the trial divisors, we find a root $x = 12.708203932$, which is correct to the last figure.

To approximate to the remainder of the other root, we take the transformation at which the roots separated, and find by

trial that 8 is the greatest figure which does not cause a loss of two variations of sign. We proceed as follows:

$$\begin{array}{llll}
5|01|98 & -456144|2 & +171616 & \underline{|.0000869364} \\
 & \underline{401584|} & \underline{-436482} & \\
 & -54560|2 & -264862 & \\
 & \underline{401584} & \underline{226286} & \\
 & 34702|4 & -38576 & \\
 & \underline{3012|} & \underline{37059} & \\
 & 37714|4 & -1517 & \\
 & \underline{3012} & \underline{1249} & \\
 & 4072|6 & -268 & \\
 & \underline{45|1} & \underline{250} & \\
 & 4117|7 & -18 & \\
 & \underline{45} & & \\
 & 4|1|6|3 & & 
\end{array}$$

Adding these figures to those already obtained, we have as a second root $x = 12.7082869364..$, which is correct to the nearest decimal.

Ex. 2. The equation,

$$7x^5 + 391x^4 + 5376x^3 - 2344x^2 + 336x - 16 = 0,$$

has three roots between 0 and 1. Before proceeding with the solution, as on the two following pages, we multiply the roots by 10, so as to dispense with the decimal points. We find $\frac{160 \times 3}{336} = 1+$, and $\frac{2244}{537 \times 3} = 1+$. We transform by 1, and in the first transformation find $\frac{424 \times 3}{300} = 4+$, and $\frac{707}{55 \times 3} = 4+$, showing that the three roots still coincide. In the second transformation we find $\frac{729 \times 3}{938} = 2+$, and $\frac{399}{55 \times 3} = 2+$, which figure must still be common to the three roots. But in the third transformation we find $\frac{882 \times 3}{1404} = 1+$, while $\frac{631}{55+3} = 3+$, showing that the roots no longer coincide. Upon trying 1, we may infer from the effect upon the final

| | | | |
|---|---|---|---|
| 7 | +3910 | +537600 | —2344000 |
| | 3917 | 3917 | 541517 |
| | 3924 | 541517 | —1802483 |
| | 3931 | 3924 | 545441 |
| | 3938 | 545441 | —1257042 |
| | 39450 [1] | 3931 | 549372 |
| | 39478 | 549372 | —707670000 [1] |
| | 39506 | 3938 | 221955648 |
| | 39534 | 55331000 [1] | —485714352 |
| | 39562 | 157912 | 222587744 |
| | 395900 [2] | 55488912 | —263126608 |
| | 395914 | 158024 | 223220288 |
| | 395928 | 55646936 | —39906320000 [2] |
| | 395942 | 158136 | 11194247656 |
| | 395956 | 55805072 | —28712072344 |
| | 39\|5970 [3] | 158248 | 11195831368 |
| | 3 | 5596332000 [2] | —17516240976 |
| | | 791828 | 11197415136 |
| | | 5597123828 | —63188258\|40 [3] |
| | | 791856 | 5599539\|0 |
| | | 5597915684 | —57588719\|4 |
| | | 791884 | 5599578\|6 |
| | | 5598707568 | —51989140\|8 |
| | | 791912 | 5599618\|2 |
| | | 5599499\|470 [3] | —463895\|22 [4] |
| | | 39\|6 | 16798\|97 |
| | | 5599539\|0 | —447096\|2 |
| | | 39\|6 | 16799\| |
| | | 5599578\|6 | —430297\| |
| | | 39\|6 | 16799\| |
| | | 5599618\|2 | —4134\|98 [5] |
| | | 39\|6 | 28\|99 |
| | | 5\|599\|658 | —4106\|98 |
| | | 5 4 | 28\| |
| | | | —4078\|98 |
| | | | 28\| |
| | | | —40\|51 |
| | | | 6 |

```
+3360000
—1802482
 1557517
—1257042
  3004750000 1
—1942857408
 1061892592
—1052506432
    93861600000 2
  —57424144688
   36437455312
  —35032481942
     140497337|0 3
    —57588719|4
      82908617|6
    —51989140|8
      3091947|6  4
    —1341288|6
      1750659|0
    —1290891|
      45976|8   5
     —20534|9
     —25441|9
      20394|9
     —504|7   6
      243|1
      261|6
      242|
       19
```

```
—1600000 |.1421356
 1557517
—4248300000 1
  4247570368
    —72963200000 2
     72874910624
       —88289376 3
        82908618
        —5380758 4
         5251977
         —128781 5
          127210
           —1571 6
            1570
              —1
```

term that this is a correct figure either of a single root, or of two roots. Upon completing the transformation, we find that the 1 is a figure of a pair of roots, for $\frac{538 \times 2}{309} = 3+$ and also $\frac{309}{46 \times 2} = 3+$. In the next transformations, till the final column is exhausted, we find both expressions, $-\frac{2C'_0}{C'_1}$ and $-\frac{C'_1}{2C'_2}$, continue to agree in the figures indicated. We may therefore suspect that the roots are equal. How we may

| | | | | | |
|---|---|---|---|---|---|
| 39\|59 | +5599499\|48 | −63188258\|4 | +140497337 | −88289376 | .000857142 |
| | 316\|72 | 44798528\|8 | −147117837 | −52964000 | |
| | 5599816\|1 | −18389729\|6 | −6620500 | −141253376 | |
| | 316\|7 | 44801062\|4 | 211290662 | 120838911 | |
| | 5600132\|8 | 26411332\|8 | 20467016\|2 | −20414465 | |
| | 316\|7 | 44803596\|0 | 3700766 | 19998034 | |
| | 5600449\|5 | 712149\|29 | 24167782\|2 | −416331 | |
| | 316\|7 | 28003\|8 | 3840784 | 291395 | |
| | 5600\|77 | 740153\|1 | 2800856\|6 | −124936 | |
| | | 28003\|8 | 56005\|6 | 116604 | |
| | | 768156\|8 | 2856862 | −8332 | |
| | | 28003\|8 | 56280 | 5831 | |
| | | 7961\|60 | 291314\|2 | −2501 | |
| | | 39\|2 | 81 | | |
| | | 8000\|8 | 291395 | | |
| | | 39\|2 | 81 | | |
| | | 8040\|0 | 29147\|6 | | |
| | | 39 | 3 | | |
| | | 80\|79 | 29151 | | |
| | | | 3 | | |
| | | | 2915\|4 | | |

verify this inference will be hereafter adverted to (187). To compute the remaining figures of the third root, whose leading figures were .142, we take the transformation at which the roots separated, and finding by trial the next figure 8, we proceed by the contracted process to find some more figures, the trial divisor indicating the remaining figures correctly. We thus find the third root, $x = .14285714\ldots$

**177.** These examples show that Horner's Method is quite equal to the separation of, and approximation to, roots, however closely coinciding in their leading figures. Such equal, or nearly equal, roots, which have usually been regarded as presenting great difficulties to the calculator, may really be approximated to with greater facility than the others, since, guided by the trial expressions A and B, we are enabled to calculate a part of several roots simultaneously as far as they coincide, and then pursue the approximation with contracted coefficients after the single roots, or groups of roots, diverge. For practice we subjoin a few equations, each having a pair of roots lying between 0 and 1, and coinciding for several figures. When, as in these examples, there are only two such roots in a given interval, the separation is easily effected by means of the trial expressions $-\dfrac{2C'_0}{C'_1}$ and $-\dfrac{C'_1}{2C'_2}$, which will suggest the same figures till the roots separate. At that point, we find by trial the correct leading figure for each, and after transforming, carry on the approximation to each by means of the ordinary trial divisor.

EXERCISES.

1. $x^4 + 152x^3 + 5101x^2 - 6780x + 2210 = 0.$
2. $x^4 + 214x^3 + 10135x^2 - 8508x + 1770 = 0.$
3. $x^4 + 284x^3 + 19171x^2 - 28530x + 10500 = 0.$
4. $x^4 + 218x^3 + 10474x^2 - 17808x + 7440 = 0.$
5. $x^4 + 188x^3 + 8651x^2 - 7900x + 1786 = 0.$
6. $x^5 + 40x^4 + 575x^3 + 1588x^2 - 848x + 104 = 0.$
7. $x^5 + 47x^4 + 823x^3 + 4964x^2 - 1454x + 104 = 0.$
8. $x^5 + 63x^4 + 500x^3 + 1334x^2 - 729x + 91 = 0.$
9. $x^5 + 93x^4 + 233x^3 + 148x^2 - 300x + 80 = 0.$
10. $x^5 + 173x^4 + 2356x^3 + 10468x^2 - 14101x + 4183 = 0.$

# CHAPTER X.

## ANALYSIS OF EQUATIONS BY FOURIER'S THEOREM.

**178.** The method of approximation explained in the preceding chapter requires, as a necessary preliminary, that the situations of the roots have been ascertained by a previous analysis. Sturm's Method is, indeed, fully adequate to the analysis of any numerical equation, but is at the same time excessively laborious in its practical application. The object of the present chapter is to show how the necessary information regarding the roots may be obtained independently of Sturm's Theorem.

**179.** PROP. I.—FOURIER'S THEOREM.*

*Let $f(x) = 0$ be an equation of the $n^{th}$ degree, then if $a$ and $b$, of which $a < b$, be successively substituted for $x$ in the series of functions $f(x)$, $f_1(x)$, $f_2(x)$, .... $f_n(x)$, consisting of $f(x)$ and its successive derived functions, the excess in the number of variations of sign in the series when $x = a$, over the number when $x = b$, will not be less than the number of real roots of $f(x) = 0$ that lie between $a$ and $b$.*

By Art. 8, no change can take place in the sign of any of the above functions except when $x$ increases through a value that makes that function vanish.

It has already been proved (148, III.) that if a function $f(x)$ vanish when $x = c$, then $f(x)$ and its first derived function $f_1(x)$ have contrary signs when $x$ is indefinitely smaller than $c$, and the same sign when $x$ is indefinitely greater than $c$. This relation must evidently hold good with regard to $f_1(x)$

* The theorem is usually named after Fourier, though first published by Budan in 1807. There is evidence, however, that Fourier had developed the theorem in MS. as early as 1797.

and its first derived function $f_2(x)$, and, generally, with regard to any function $f_r(x)$ and its first derived function $f_{r+1}(x)$.

We have two cases to consider:

I. Suppose that for some value between $a$ and $b$, as $x = c$, the $r$ consecutive functions $f(x)$, $f_1(x)$, $f_2(x)$, .... $f_{r-1}(x)$, vanish, but $f_r(x)$ does not.

From what has been said above, these $r + 1$ functions will, when $x$ is indefinitely smaller than $c$, present $r$ variations of sign, and, when $x$ is indefinitely greater than $c$, no variation.

Hence, *as $x$ increases through a root $c$, that occurs $r$ times in $f(x) = 0$, $r$ variations of sign are lost in the series of functions, but none gained.*

II. Suppose that when $x = c$ the $m$ derived functions $f_r(x)$ to $f_{r+m-1}(x)$ vanish, but $f_{r-1}(x)$ and $f_{r+m}(x)$ do not.

Then in the series of $m + 1$ functions $f_r(x)$, .... $f_{r+m-1}(x)$, $f_{r+m}(x)$, there will be $m$ variations of sign when $x$ is indefinitely smaller than $c$, and $f_r(x)$ will have the same sign as $f_{r+m}(x)$ if $m$ is an even number, and the contrary sign if $m$ is an odd number. Therefore the series of $m + 2$ functions,

$$f_{r-1}(x),\ f_r(x),\ \ldots.\ f_{r+m-1}(x),\ f_{r+m}(x),$$

will, if the extreme functions, $f_{r-1}(x)$ and $f_{r+m}(x)$, have the *same* sign, present $m$ or $m + 1$ variations of sign (according as $m$ is even or odd) when $x$ is indefinitely smaller than $c$, and no variation when $x$ is indefinitely greater than $c$; while if $f_{r-1}(x)$ and $f_{r+m}(x)$ have *contrary* signs, there will be $m + 1$ or $m$ variations (according as $m$ is even or odd), when $x$ is indefinitely smaller than $c$, and one variation, when $x$ is indefinitely greater than $c$.

Hence, *as $x$ increases through a value that makes $m$ consecutive derived functions vanish, $m$ variations are lost when $m$ is an even number, and $m \pm 1$ when $m$ is an odd number;* that is, the number of variations lost in this way is always an even number, and none is gained.

Thus, as $x$ increases from $a$ to $b$, at least as many variations are lost (I.) as there are real roots in the interval; also if there are more variations lost than there are real roots passed, the excess must be an even number (II.).

**180.** Cor. 1.—*If the coefficients of an equation $f(x) = 0$, present $m$ more variations of sign than those of the transformed equation $f(a + x') = 0$, then the number of roots passed in the transformation is $m$, or $m$ minus some even number.*

For the coefficients of $f(x)$ are the values of the functions,

$$\frac{1}{\lfloor n}f_n(x), \ldots . \frac{1}{\lfloor r}f_r(x), \ldots . \tfrac{1}{2}f_2(x),\ f_1(x),\ f(x),$$

when 0 is put for $x$, and the coefficients of $f(a+x')$ are the values of the same functions when $a$ is put for $x$. If, then, there are $m$ variations fewer in the signs of the coefficients of $f(a+x')$ as compared with those of $f(x)$, this is the same as a loss of $m$ variations in the signs of the series of functions $f_n(x) \ldots . f_r(x) \ldots . f(x)$, as $x$ increases from 0 to $a$, thus indicating $m$ roots in the interval, or $m$ minus some even number.

**181.** Cor. 2.—*If no variation is lost in the signs of the coefficients of a transformed equation, as compared with the preceding one, there is no root in the interval.*

**182.** Cor. 3.—*If any odd number of variations is lost, there is some odd number of roots in the interval; whether more than one, we cannot say.*

**183.** Cor. 4.—*If any even number of variations is lost, there is either no root, or some even number of roots in the interval.*

**184.** Cor. 5.—*If an equation have an odd number, $p$, of consecutive zero coefficients, these zeros indicate $p \pm 1$ imaginary roots, according as the zeros occur between similar or contrary signs; if there be an even number, $q$, of consecutive zero coefficients, these, whether they occur between similar or contrary signs, indicate $q$ imaginary roots.*

This follows from Art. 179, II; for if we imagine the proposed equation transformed into another, having as roots those of the proposed each increased by an indefinitely small quantity, the zeros would be replaced by numbers having alternately positive and negative signs, while if the roots were diminished

by an indefinitely small quantity, the zeros would be replaced by numbers having the same sign, and, as by supposition, the zeros occur between terms that are not zero, variations have been lost that do not indicate real roots.

**185.** General directions for the application of this theorem to the determination of the intervals in which positive roots may lie, may be given as follows:

(1). Transform successively by 10, 100, 1000, &c., as far as necessary, so as to ascertain how many roots may be found in the intervals [0, 10], [10, 100], &c. This step may in many cases be omitted, as a cursory application of the rules for limits (75, 76) may show that there can be no root so great as 10, &c.

(2). Proceed by successive transformations by 1 to find between what units the roots in the interval [0, 10], if any, lie, singly or in groups; for those in the interval [10, 100] we proceed by 10's; for those in the interval [100, 1000], by 100's.

(3). Roots found to lie singly between consecutive units, or tens, or hundreds, may be regarded as ready for the application of Horner's Method, since we have thus the leading figure of a root lying singly. But if more than one root is indicated between consecutive tens, we proceed to subdivide the interval, by successive transformations by 1, till we find between what successive integers the roots lie singly or in pairs; we proceed similarly when several roots are indicated between consecutive hundreds.

**186.** By this process we are thus enabled either to obtain the initial figure of each root singly, or arrive at intervals of consecutive units between which two or more roots may lie. Proceeding, in this case, to develop the roots according to the precepts given in Art. 176, we may find: (1) by their separation after a few figures, that the roots in question are real and unequal (see examples in 176); (2) by their continuing to concur to seven or more decimals, we may infer that the roots are equal, and proceed to verify the inference in the manner shown below; (3) the roots are imaginary, of which we shall be made aware by tests presently to be explained.

**187.** We shall illustrate each of these cases, taking for the first example the difficult equation, already given in Art. 158.

Ex. 1. $$x^6 + 378x^5 + 38189x^4 + 492368x^3 - 572554x^2 + 213720x - 26352 = 0.$$

As it is evident by inspection that there is a root less than unity, we first transform by 1. In transforming by such numbers as 1, 10, &c., there is no effective multiplication, and the easy additions and subtractions may be performed mentally, the results only being set down as in the following example:

$$\begin{array}{lllllll|l}
1 & +378 & +38189 & +492368 & -572554 & +213720 & -\ 26352 & 1 \\
 & 379 & +38568 & +530936 & -\ 41618 & +172102 & +145750' & \\
 & 380 & +38948 & +569884 & & & &
\end{array}$$

We need pursue this transformation no further, as it is evident that all the three variations of sign in the proposed equation will be lost; we thus know that there is at least one real root in the interval [0,1], possibly three.

To ascertain the intervals in which the negative roots may lie, we change the signs of alternate terms, and proceed first to transform by 10.

$$\begin{array}{lllllll|l}
1 & -378 & +38189 & -492368 & -\ 572554 & -\ 213720 & -\ 26352 & 10 \\
 & -368 & +34509 & -147278 & -2045334 & -20667060 & -206696952' & \\
 & -358 & +30929 & +162012 & -\ 425214 & -24919200' & & \\
 & -348 & +27449 & +436502 & -3939806' & & & \\
 & -338 & +24069 & +667192' & & & & \\
 & -328 & +20789' & & & & & \\
 & -318' & & & & & &
\end{array}$$

As the number of variations remains the same, we know that there is no root in the interval [0, −10]. As inspection of the transformed coefficients makes it probable that a further transformation by 10 will cause a change in the number of variations, we proceed:

$$\begin{array}{lllllll|l}
1 & -318 & +20789 & +677192 & -3939806 & -24919200 & -206696952 & 10 \\
 & -308 & +17709 & +854282 & +4603014 & +21100940 & +\ 4412448 &
\end{array}$$

We need proceed no further with this transformation; one variation is lost, and it is plain that no more can be lost, since

the second term, at least, will remain negative; a root, therefore, lies in the interval [—10, —20].

We now transform by 100, to ascertain whether more roots lie in the interval [0, —100].

$$\begin{array}{rrrrrrrl}
1 & -378 & +38189 & -\ 492368 & -\ 572554 & -\ 213720 & -\ 23652 & \underline{\lfloor 100} \\
 & -278 & +10389 & +\ 546532 & +\ 54080646 & +5407850880 & +540785061648' & \\
 & -178 & -\ 7411 & -\ 194568 & +\ 34623846 & +8870235480' & & \\
 & -\ 78 & -15211 & -1715668 & -136942954' & & & \\
 & +\ 22 & -13011 & -3016768' & & & & \\
 & +122 & -\ 811' & & & & & \\
 & +222' & & & & & &
\end{array}$$

No variations but the one due to the root in the interval [—10, —20] are lost by this transformation; no roots, therefore, lie in the interval [—20, —100].

We proceed to a further transformation by 100:

$$\begin{array}{rrrrrrrl}
1 & +222 & -\ 811 & -3016768 & -136942954 & +8870235480 & +540785061648 & \underline{\lfloor 100} \\
 & 322 & +31389 & +\ 122132 & -124729754 & -3602739920 & +180511069648 & \\
 & 422 & +73589 & +7481032 & +623373446 & + \ldots & &
\end{array}$$

We perceive already that the two variations will be lost in this transformation, indicating the presence of two roots in the interval [—100, —200]. We can now, starting from the transformation by 100, transform repeatedly by 10, or, what is sometimes preferable, halve the interval by transforming by 50. Proceeding in this manner, we find a variation is lost between the transformations by 160 and 170, then employing only the first line of transformations, we easily find the remaining root is situated between 190 and 200. We have thus ascertained the presence of single real roots in the intervals [—10, —20], [—160, —170], [—190, —200]. Of the three roots indicated in the interval [0, 1] we know that one, at least, must be real. As these roots are quite small as compared with the others, we may expect the trial expressions ($A$) and ($B$) for three roots (175) to suggest the same leading figures, which they do, $\dfrac{263\times 3}{2137} = .3+$ and $\dfrac{572}{492\times 3} = .3+$. Proceeding by the aid of these trial divisors to develop the roots, as in the examples in Art. 176, we find the three roots concur in the first four figures; at the fourth transformation a discrepancy in the

figures suggested by the two trial expressions warns us that the roots no longer concur. By the loss of one variation, when we try 1 as the trial figure, we find the root separated is the smallest of the three, the remaining two we find concur in one figure more, then separate, showing that they too are real.

Ex. 2. $7x^5 + 391x^4 + 5376x^3 - 2344x^2 + 336x - 16 = 0.$

$$\begin{array}{rrrrrr|l}
7 & +391 & +5376 & -2344 & +336 & -16 & 1 \\
 & 398 & +5774 & +3430 & +3766 & +3750 &
\end{array}$$

Before the completion of the first line of transformation it is apparent that all three variations of the proposed will be lost in the transformed equation, thus indicating three roots in the interval [0, 1]. To ascertain the situations of the remaining roots, we change the signs of alternate terms, and proceed thus :

$$\begin{array}{rrrrrr|l}
7 & -391 & +5376 & +2344 & +336 & +16 & 10 \\
 & -321 & +2166 & +24004 & +240376 & +2403776 & \\
 & -251 & -344 & +20564 & +446016 & & \\
 & -181 & -2154 & -976 & & & \\
 & -111 & -3264 & & & & \\
 & -41 & & & & &
\end{array}$$

No variations lost ; we transform again by 10

$$\begin{array}{rrrrrr|l}
7 & -41 & -3264 & -976 & +446016 & +2403776 & 10 \\
 & +29 & -2974 & -30716 & +138856 & +3792336 & \\
 & +99 & -1984 & -50556 & -366704 & & \\
 & +169 & -294 & -53496 & & & \\
 & +239 & +2096 & & & & \\
 & +309 & & & & &
\end{array}$$

No variations lost ; we transform again

$$\begin{array}{rrrrrr|l}
7 & +309 & +2096 & -53496 & -366704 & +3792336 & 10 \\
 & 379 & +5886 & +5364 & -313064 & +661696 & \\
 & 449 & +10376 & +109124 & +778176 & & \\
 & 519 & +15566 & +264784 & & &
\end{array}$$

It was evident before the completion of the second line of this transformation that all the coefficients would be positive,

the loss of two variations indicating two roots in the interval [—20, —30]. But, from a comparison of the final term of the last transformation with that of the preceding one, the roots would appear to lie much nearer to 30 than 20. It was therefore worth while to complete the third line of the transformation so as to obtain the coefficient $C'_2$: we then find that $\frac{-2C'_0}{C'_1}$, or $\frac{-1322}{778}$, $= -(1+)$, and also $\frac{-C'_1}{2C'_2} = -(1+)$, from which we infer that 30 is *greater* than both roots by $1+$, *i. e.*, the roots lie in the interval [—28, —29], which inference is verified by trial.* Thus the roots of the proposed equation lie, three in the interval [0, 1], two in [—28, —29].

In Art. 176 the three roots in the interval [0, 1] we found to concur to three places of decimals, at the fourth figure one separating. The remaining two were found, even by the contracted process, to concur to seven places of decimals, a fact suggestive of the possibility of their being equal.

Now, by Art. 95, if an equation have two incommensurable equal roots, it must have another pair, and be divisible by a quadratic factor having commensurable coefficients. If, therefore, as in this example, besides the pair of roots supposed to be equal, another pair of roots is indicated in another interval, these, if the roots are really equal, must be, what may be termed, the conjugates of the supposed equal roots, and have their decimal parts the same, if the two pairs are of opposite signs, or complementary, if of the same sign; also the product of a conjugate pair must be integral. Here we have the supposed equal roots each $= .1421356..$; the other pair, if the supposition is correct, must each $= -28.1421356..$; the sum of a pair of these, with changed sign, is 28, their product is $-4$ approximatively. We try whether $x^2 + 28x - 4$ will divide the first member of the equation; as it does so, we know that there are two pairs of equal roots.

---

* Whenever, in the course of our search, we find a root, or roots, indicated in a certain interval, we may, as in this example, employ the coefficients of that transformation which has the smaller final term to guide us to the next figure, the sign of the suggested figure showing whether it is to be added to the smaller, or subtracted from the greater, of the numbers between which the roots are indicated. Even when the final terms show no decided difference, the numbers suggested will often be complementary, or nearly so, thus showing how far we may depend on them.

Similarly, if $r$ roots $a + \delta$, where $\delta$ is the decimal part, are found to concur to so many places of decimals as to render it probable that they are equal, and there is another interval containing $s$ possibly equal roots with an integral part $b$, then if $(a+\delta)(b+\delta) = c$, an integer, approximatively, there is *prima facie* evidence that some of the roots are equal, in which case $x^2 - (a+b)x + c$* will divide the first member of the equation as many times as there are pairs of conjugate roots.

Ex. 3. $x^5 - x^4 - 15x^3 - 5x^2 + 53x + 51 = 0.$

We proceed as in the preceding examples to ascertain the interval in which roots may lie; as the method of procedure has been sufficiently exemplified, we shall give merely the results.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| [0]. | 1 | $-1$ | $-15$ | $-5$ | $+53$ | $+51$, | two variations. |
| [1]. | 1 | $+4$ | $-9$ | $-46$ | $-1$ | $+84$, | " " |
| [2]. | 1 | $+9$ | $+17$ | $-39$ | $-99$ | $+33$, | " " |
| [3]. | 1 | $+14$ | $+63$ | $+76$ | $-85$ | $-78$, | one variation. |
| [4]. | 1 | $+19$ | $+129$ | $+359$ | $+317$ | $-9$, | " " |

It is evident that the remaining variation will disappear in the next transformation; the positive roots are therefore real, and lie in the intervals [2, 3], [4, 5]. To find the places of the negative roots, we change the signs of alternate terms, and proceeding in the usual way, obtain the following results:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| [0]. | 1 | $+1$ | $-15$ | $+5$ | $+53$ | $-51$, | three variations. |
| [1]. | 1 | $+6$ | $-1$ | $-24$ | $+27$ | $-6$, | " " |
| [2]. | 1 | $+11$ | $+33$ | $+19$ | $+5$ | $+3$, | no variation. |

We infer from the loss of three variations that there is, at least, one real root in the interval [—1, —2], possibly three. The fact that two of the roots are imaginary is shown at once by the test we are about to establish.

**188.** DEF.—In Art. 66 it was shown that if $C_r$ be the coefficient of $x^r$ in an equation $f(x) = 0$, then

$$C_r^2 - 2C_{r+1}\cdot C_{r-1} + 2C_{r+2}\cdot C_{r-2} - 2C_{r+3}\cdot C_{r-3} + \text{\&c.},$$

will [taken negatively if $x^r$ occur in an even term in $f(x)$]

* Or $x^2 - (a+b+1)x + c$, when $a$ and $b$ have like signs.

be the coefficient of $(x^2)^r$ in $F(x^2) = 0$, the equation that has for roots the squares of the roots of $f(x) = 0$: this expression, or its numerical value, we shall refer to as a *coefficient function* of $f(x)$.

**189.** Prop. II.—*There are at least as many imaginary roots in an equation $f(x) = 0$ as there are variations in the signs of its coefficient functions.*

For, since $F(x^2) = 0$ must have as many positive real roots as $f(x) = 0$ has real roots, the number of real roots in $f(x) = 0$ cannot be greater than the number of variations of sign in $F(x^2)$, and consequently the number of imaginary roots cannot be less than the number of permanences of sign in $F(x^2)$. Now of the coefficient functions the first $C_n^2$ and the last $C_0^2$ are always positive, so that the number of variations in the signs of the whole series of functions must be an even number, if any, and indicate an equal number of permanences in the signs of $F(x^2)$, since the values of these expressions, with the sign of each alternate one changed, are the coefficients of $F(x^2)$; the number of imaginary roots in $f(x) = 0$ cannot, therefore, be less than the number of variations in the signs of the coefficient functions.

**190.** Cor.—*If the square of any coefficient $C_r$ be less than $C_{r+1} \cdot C_{r-1}$, the product of the contiguous coefficients, a pair of imaginary roots is thus indicated.*

If $f(x) = C_n x^n + \ldots C_{r+1} x^{r+1} + C_r x^r + C_{r-1} x^{r-1} + \ldots C_0 = 0$, then the $(r-1)^{th}$ derived function, when divided by $\underline{|r-1}$, will have for its last three terms $\dfrac{r(r+1)}{2} C_{r+1} x^2 + r C_r x + C_{r-1}$; there will be a pair of imaginary roots in $f_{r-1}(x) = 0$, and therefore (88), in $f(x) = 0$, if $r^2 C_r^2 < r(r+1) C_{r+1} \cdot C_{r-1}$, or $C_r^2 < \dfrac{r+1}{r} C_{r+1} \cdot C_{r-1}$, and, *a fortiori*, if $C_r^2 < C_{r+1} \cdot C_{r-1}$.

This corollary will often be found sufficient to determine the character of the roots in a doubtful interval; in a nice case it may be important not to neglect the factor $\dfrac{r+1}{r}$, especially when $r$ is a small number.

**191.** The test provided by the above proposition is so simple in its application that a glance at the coefficients of an equation will often suffice to show that some, if not all, of its roots are imaginary.

Ex. 1. $x^5 + x^4 + x^3 - 2x^2 + 2x - 1 = 0.$

Here the coefficient functions, as found by the rule, are,

$$1 \quad -1 \quad +9 \quad -2 \quad \pm 0 \quad +1, \text{ four variations;}$$

there are accordingly four imaginary roots in the equation.

It is obviously not necessary to obtain the actual values as we have done in this example, the signs alone being important. The first and the last are always positive, and the rest are usually obvious to inspection.

Ex. 2. $x^7 - 2x^5 - 3x^3 + 4x^2 - 5x + 6 = 0.$

When, as in this example, the equation is not complete, we must supply the place of the absent terms by zeros before proceeding to determine the signs of the coefficient functions. We here obtain the series of signs, $+ + - - + + - - +$, and see that there are at least four imaginary roots.

Ex. 3. $5x^6 - 19x^5 + 43x^4 - 75x^3 + 438x^2 - 412x + 253 = 0.$

From this we obtain the series of signs, $+ - + - + - +$, and see that the roots are all imaginary.

In the last example of Art. 187 we arrived at a transformation,

$$1 \quad +11 \quad +33 \quad +19 \quad +5 \quad +3,$$

where two variations had been lost, the corollary above given shows that the two roots in the interval are imaginary since $5^2 < 19 \times 3$, thus completing the analysis of the equation. Having showed that variations in the signs of the coefficient functions indicate imaginary roots, we have now to prove the converse of that proposition.

**192.** Prop. III.—*If an equation $f(x) = 0$ have a pair of imaginary roots $a \pm \beta\sqrt{-1}$, there are certain limits $\lambda$ and $\mu$, one less than $a$, the other greater, such that, if the equation be transformed to $f(y+x') = 0$, where $y$ is any*

*real quantity between* $\lambda$ *and* $\mu$,* *one at least of the coefficient functions will be negative.*

(1). If $f(x) = 0$ be transformed into $f(a+x') = 0$, in the latter equation the roots $a \pm \beta\sqrt{-1}$ will be reduced to $\pm\beta\sqrt{-1}$, which roots in $F(\overline{a+x'^2}) = 0$ will become a pair of negative real roots, $-\beta^2$, $-\beta^2$. These (53) must produce a pair of permanences in the signs of the coefficients of $F(\overline{a+x'^2})$, corresponding to which must be a pair of variations in the signs of the coefficient functions of $f(a+x')$. Hence there is a transformed equation which has at least one negative function corresponding to the supposed pair of imaginary roots.

(2). Supposing $f(x)$ to be of the $n^{th}$ degree, then

$$f(y+x') = C_n x'^n + \frac{1}{\underline{|n-1}} f_{n-1}(y) x'^{n-1} + \ldots \tfrac{1}{2} f_2(y) x'^2 + f_1(y) x' + f(y).$$

Forming the coefficient functions according to 188, we have

$$C_n^2;\ \left[\frac{1}{\underline{|n-1}} f_{n-1}(y)\right]^2 - 2C_n \cdot \frac{1}{\underline{|n-2}} f_{n-2}(y); \ldots [f_1(y)]^2 - f_2(y) f(y);$$
$$[f(y)]^2.$$

Of these functions, which are all of even degree, the first and last cannot be negative, and of the rest none can become negative if all the roots of $f(x) = 0$ are real (189). But, if $f(x) = 0$ have a pair of imaginary roots $a \pm \beta\sqrt{-1}$, it has been proved that some one of these functions, say $[f_1(y)]^2 - f_2(y) f(y)$, becomes negative, at least when $y = a$. This function must therefore become zero for some value of $y$ greater than $a$, as $\mu$, and also, being of even degree, for some value $\lambda$, less than $a$. Thus, if the roots of $f(x) = 0$ be diminished by any quantity between $\lambda$ and $\mu$, at least one of the series of coefficient functions in the transformed equation will be negative.

**193.** We say, at least one, for several consecutive functions may be negative for the same value of $y$, and these, in like manner, continue negative for values of $y$ between other limits

---

* The signs of these limits may of course be different, and one of the admissible values of $y$ be 0, as in the examples in Art. 191.

$[\lambda', \mu']$, &c. If these negative functions were not consecutive, there would be more than one pair of variations, and therefore more than one pair of imaginary roots indicated.

**194.** By considering the subject in the following manner, we obtain some insight into the relations existing between the limits $[\lambda, \mu]$ and the roots $a \pm \beta\sqrt{-1}$.

Corresponding to a pair of roots $a \pm \beta\sqrt{-1}$ in $f(x) = 0$ is a factor $x^2 - 2ax + (a^2 + \beta^2)$ in $f(x)$, and a factor $x^4 - 2(a^2 - \beta^2)x^2 + (a^2 + \beta^2)^2$ in $F(x^2)$.

As long as $a'^2 > \beta^2$ (where $a' = a - y$), the real part of the imaginary roots $(a' \pm \beta\sqrt{-1})^2$ in $F(\overline{y+x'}^2) = 0$ remains positive; but, since as $y$ increases in $f(y+x') = 0$, $a'$ becomes smaller while $\beta$ remains constant, $a'^2 - \beta^2$ first becomes negative; and, when $y = x$, the factor in $F(\overline{y+x'}^2) = 0$ becomes $x^4 + 2\beta^2x^2 + \beta^4$, representing, as we have seen, a pair of real negative roots. As $y$ still increases, $a'^2$, after passing through zero, increases till it again reaches a value at which the coefficient $-2(a'^2 - \beta^2)$ no longer causes permanences in the signs of $F(\overline{y+x'}^2)$. Thus, as $-2(a'^2 - \beta^2)$ is positive only when $a'$ has a value between $+\beta$ and $-\beta$, it would appear that the difference between $\lambda$ and $\mu$ must generally be somewhat less than $2\beta$, or $\lambda$ and $\mu$ lie between $a - \beta$ and $a + \beta$.

Or, suppose $F(\overline{y+x'}^2) = \phi(z^2)[z^4 - 2(a'^2 - \beta^2)z^2 + (a'^2 + \beta^2)^2,]$ where $z = y + x'$, and $\phi(z^2)$ is the function involving the squares of the real roots, and of those imaginary roots within whose limits we shall first suppose $y$ does not here fall. The signs of $\phi(z^2)$ will present a series of variations, say,

$$+ \; - \; + \; - \; + \; - \; + \; - \; +.$$

Now, if in the factor $z^4 - 2(a'^2 - \beta^2)z^2 + (a'^2 + \beta^2)^2$, we have $a'^2 = \beta^2$, the series of signs in that factor will be $+\;0\;+$, and the multiplication by that factor will merely add two variations to those of $\phi(z^2)$, since like signs will come under like: thus in this case $y = a \pm \beta$ does not fall within the limits of $a \pm \beta\sqrt{-1}$. But if the signs of $\phi(z^2)$ do present a pair of permanences, let the series of signs be,

$$+ \; - \; + \; - \; + \; - \; + \; + \; +,$$

then the multiplication by the factor presenting the series of signs $+0+$ will produce the series $+-+-+-+\pm+++$, which may have four permanences: thus $y = a \pm \beta$ may fall within the limits of the roots $a \pm \beta\sqrt{-1}$, if at the same time it be within the limits of another pair of imaginary roots.

**195.** Hence, if there be two pairs of imaginary roots, $a \pm \beta\sqrt{-1}$ and $\gamma \pm \delta\sqrt{-1}$, then of the four quantities, $a-\beta$, $a+\beta$, $\gamma-\delta$, $\gamma+\delta$, (1) if $a+\beta =$ or $< \gamma-\delta$, the two pairs of roots will indicate their presence by producing pairs of variations in the signs of the coefficient functions of *different* transformations: (2) if both $a-\beta$ and $a+\beta$ lie between $\gamma-\delta$ and $\gamma+\delta$, the two pairs of roots will indicate their presence by four variations in the coefficient functions of *some one* transformation; this includes the cases where $a = \gamma$, whether $\beta = \delta$ or not: (3) if $a+\beta$ lies between $\gamma-\delta$ and $\gamma+\delta$, the two pairs of roots may or may not give indication of their presence in the same transformation.

**196.** If the difference between $\lambda$ and $\mu$ be not less than unity, whenever, indeed, $\lambda$ and $\mu$ do not both lie between consecutive integers, the character of imaginary roots is easily determined. For when in the course of our transformations to discover in what intervals roots may lie, we have narrowed down a doubtful interval to between two consecutive integers, one or other of these integers must, in the supposed case, fall between the limits, and one or other of the transformations have a negative coefficient function.* We shall first illustrate this, the more common case, by a few examples, leaving the more difficult cases for future consideration.

---

* The function $[f_1(y)]^2 - f_2(y)f(y)$ equated to zero, or $\phi(y) = 0$, of which $\lambda$ and $\mu$ are roots, may be regarded as a limiting equation to the real parts of imaginary roots of $f(x) = 0$, just as $f_1(x) = 0$ is to the real roots. The first derived function of $\phi(y)$ equated to zero, or $f_2(y)f_1(y) - f_3(y)f(y) = 0$, must have a real root between $\lambda$ and $\mu$. We could, therefore, if there were not easier methods, always thus determine a value of $y$ which would make $f(y+x')$ present variations in the signs of its coefficient functions, if the roots in the doubtful interval are imaginary.

Ex. 1. Let the equation proposed for analysis be,

$$4x^5 - 28x^4 - 39x^3 + 504x^2 - 239x - 1855 = 0.$$

We see that there is probably no root greater than 10; we therefore, transforming successively by 1, obtain the following results:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| [0]. | 4 | −28 | − 39 | +504 | −239 | −1855; | three variations. |
| [1]. | 4 | − 8 | −111 | +259 | +560 | −1653; | " " |
| [2]. | 4 | +12 | −103 | − 82 | +733 | − 949; | " " |
| [3]. | 4 | +32 | − 15 | −279 | +328 | − 385; | " " |
| [4]. | 4 | +52 | +153 | − 92 | −127 | − 315; | one variation. |

Here two variations are lost, and since obviously $\overline{127}^2 < 184 \times 315$, the roots indicated are imaginary. The remaining positive root must be real, and its interval [5, 6] is found by finding what number will make the final term positive.

To find the intervals of the negative roots, we change the signs of alternate terms, and transforming successively by 1, obtain the following results:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| [0]. | 4 | +28 | − 39 | −504 | − 239 | +1855; | two variations. |
| [1]. | 4 | +48 | +113 | −413 | −1232 | +1105; | " " |
| [2]. | 4 | +68 | +345 | +254 | −1507 | − 375; | one variation. |

The loss of a variation here shows that the negative roots are real, and one lies in the interval [—1, —2]. The remaining root is found, as above, to lie in the interval [—3, —4]. The proposed equation has therefore three real roots lying in the intervals [—3, —4], [—1, —2], and [5, 6], and two imaginary roots, the real part of which is not far from 4.

Ex. 2. Let the equation proposed for analysis be,

$$x^6 - 6x^5 + 40x^3 + 60x^2 - x - 1 = 0.$$

Proceeding as above, we obtain the results:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| [0]. | 1 | − 6 | + 0 | +40 | + 60 | − 1 | − 1; | three var. |
| [1]. | 1 | + 0 | −15 | + 0 | +135 | +215 | + 93; | two var. |
| [2]. | 1 | + 6 | + 0 | −40 | + 60 | +431 | +429; | " " |
| [3]. | 1 | +12 | +45 | +40 | + 15 | +467 | +887; | no var. |

The loss of a variation indicated a real root in the interval [0, 1]; the loss of two variations in the interval [2, 3] leads us to apply the test for imaginary roots, and as, obviously, $\overline{15}^2 < 40 \times 467$, the roots indicated are imaginary. To find the negative roots, changing alternate signs, we have:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| [0]. | 1 | + 6 | + 0 | −40 | +60 | + 1 | − 1; | three variations. |
| [1]. | 1 | +12 | +51 | +68 | +63 | +73 | +37; | no variation. |

Here three roots are indicated in the interval [0, −1], of which one must be real; but, since $\overline{63}^2 < 68 \times 73$, the other two must be imaginary. The equation, therefore, has but two real roots, one in each of the intervals [−1, 0], [0, 1].

**197.** In the preceding examples the presence of imaginary roots, where they occur, has been detected with facility, as will always be the case except when $\beta$, the coefficient of the imaginary sign, is so small that the limits $[\lambda, \mu]$ fall between consecutive integers.

When we arrive at a doubtful interval between two consecutive integers,* that is, one where the coefficients of neither transformation give indications of imaginary roots, the roots in the interval may be (1) real roots, equal or nearly equal, in which case we can, as shown in (187), by the aid of the proper trial expressions, determine them to any degree of accuracy desired; (2) the roots, or some even number of them, may be imaginary, having $\beta$ small. Now the smaller $\beta$ is, the more closely do the imaginary roots approach to being equal real roots, and the proper trial expressions will guide to successive figures of the real part of the roots, till a discrepancy in the suggested figures warns us that either a group of roots is about to separate, or that we have come within the narrow limits where the coefficient functions give indications of the roots being imaginary. In the case of a doubtful interval, therefore, we proceed according to the rules in Art. 176, and when a discrepancy in the trial expressions occurs, apply the test for imaginary roots.

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* One of which, obviously, must be within .5 of the roots, or real part of the imaginary roots.

Ex. 1. Let the proposed equation be,

$$4x^7 - 6x^6 - 7x^5 + 8x^4 + 7x^3 - 23x^2 - 22x - 5 = 0.$$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| [0]. | 4 | − 6 | − 7 | + 8 | + 7 | −23 | −22 | − 5; | three var. |
| [1]. | 4 | +22 | +41 | +23 | −11 | −30 | −58 | −44; | one var. |

Here two variations are lost; and since $\overline{30}^2 < 2(11 \times 58 + 23 \times 44)$, the roots indicated are imaginary. The remaining positive root is of course real, and its situation is easily found.

To find the negative roots, changing the signs of alternate terms, and proceeding as usual, we obtain,

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| [0]. | 4 | + 6 | − 7 | − 8 | + 7 | + 23 | −22 | +5; | four var. |
| [1]. | 4 | +34 | +113 | +187 | +165 | +100 | +42 | +8; | no var. |

Here are four variations lost, and since $\overline{165}^2 < 2(187 \times 100 - 113 \times 42)$, two at least of the roots must be imaginary. Whether the other two are so, evidently depends upon whether the real root of $f_1(x) = 0$ in the interval separates them. As $\frac{5 \times 2}{22} = .45..$, and $\frac{22}{23 \times 2} = .47..$, we may expect the roots, if real, to coincide in the two figures .46 at least; proceeding in the approximation, we find the roots agree in the four figures .4616, and separate at the fifth figure; these roots are therefore real.

Ex. 2. Let the proposed equation be,

$$x^5 - 91x^4 + 2518x^3 + 17870x^2 + 4617x - 82580 = 0.$$

Transforming several times by 10, we obtain,

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| [0]. | 1 | −91 | +2518 | +17870 | + 4617 | − 82580; | three var. |
| [10]. | 1 | −41 | − 122 | +48810 | + 803417 | + 3458590; | two var |
| [20]. | 1 | + 9 | − 762 | +30550 | +1629017 | +15941760; | " " |
| [30]. | 1 | +59 | + 598 | +23090 | +2107417 | +34814930; | no var. |

By the loss of a variation in the interval [0, 10] a real root is there indicated; the loss of two variations in the interval [20, 30] indicates two roots, which, without subdividing the interval, we see must be imaginary, since $\overline{598}^2 < 23090 \times 59$.

To find the situations of the negative roots, we change alternate signs, and, seeing that 8 is a superior limit, we transform several times by 1; thus we obtain,

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| [0]. | 1 | + 91 | +2518 | −17870 | + 4617 | +82580; | two var. |
| [1]. | 1 | + 96 | +2892 | − 9760 | −23200 | +71937; | " " |
| [2]. | 1 | +101 | +3286 | − 498 | −33655 | +41966; | " " |
| [3]. | 1 | +106 | +3700 | + 9976 | −24384 | +11201; | " " |
| [4]. | 1 | +111 | +4134 | +21722 | + 7097 | + 600; | no var. |

The roots indicated in the interval [3, 4] may be real, as neither transformation has a negative coefficient function. The roots, if real, are evidently nearer to 4 than 3; as both $\frac{-600 \times 2}{7097}$ and $\frac{-7097}{21722 \times 2}$ suggest $-.16..$, we infer that the roots are not far from 3.83... We accordingly, proceeding from the transformation by 3, transform first by .8, and again by .03, though there is a slight discrepancy in the figures suggested by the trial expressions of the transformation by .8, the one suggesting .03.., the other .028... We obtain as the three right-hand terms in this transformation 19632.7.., 67.76.., 1.058.., and as obviously $\overline{67.7}^2 < 19632 \times 2$, the roots indicated are imaginary, the real part being not far from 3.83.., and the imaginary part less than $.05\sqrt{-1}$.

Ex. 3. In the equation,

$$x^5 + 1755x^4 + 270450x^3 + 14262750x^2 - 17791875x + 5484376 = 0,$$

two variations are lost in the interval [0, 1]. Here the trial expressions $\frac{548 \times 2}{1779}$ and $\frac{1779}{1426 \times 2}$ both suggest .6. Proceeding with the approximation, as on the following page, we find the same figures suggested by both expressions in each transformation, till we have obtained .6129. At the fourth transformation the trial expressions suggest widely different figures, and as obviously $\overline{2198}^2 < 108168 \times 147 \times 2$, the roots are imaginary, having the real part about .6129, and the imaginary part about $.0001\sqrt{-1}$.

**198.** When the imaginary part is very small, as in this example, we evidently cannot obtain evidence of the imaginary

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | +17550 | +27045000 | +14262750000 | −177918750000 | +548437600000 | .6129 |
| | | 105336 | 162902016 | 86553912096 | −548189027424 | |
| | 17556 | 27150336 | 14425652016 | −91364837904 | 248572576 1 | |
| | | 105372 | 163534248 | 87535117584 | −235411034 | |
| | 17562 | 27255708 | 14589186264 | −382972032 0 1 | 13161542 2 | |
| | | 105408 | 164166696 | 147560997 9 | −11659656 | |
| | 17568 | 27361116 | 147533529 60 | −235411034 | 1501886 3 | |
| | | 105444 | 27468 32 | 147588468 | −1393718 | |
| | 17574 | 27466 560 1 | 147560997 92 | −8782256 6 2 | 108168 4 | |
| | | 1 758 | 27470 07 | 2952428 6 | | |
| | 1 7580 | 27468 318 | 147588468 | −5829828 | | |
| | 1 | 1 758 | 27471 | 2952539 | | |
| | | 27470 076 | 1476159 39 2 | −287728 9 3 | | |
| | | 1 758 | 54 94 | 132871 3 | | |
| | | 27471 834 | 1476214 33 | −154857 6 | | |
| | | 1 758 | 1476269 27 | 132873 5 | | |
| | | 27 473 | 14763 24 3 | −2198 4 4 | | |
| | | 3 2 | 24 | | | |
| | | | 14763 48 | | | |
| | | | 147 63 72 | | | |
| | | | 4 | | | |

character of the roots till we have carried out the approximation to so many decimals as will make the real part $a$ less than $\beta$, the coefficient of $\sqrt{-1}$. Yet the labor of ascertaining that a pair of roots is imaginary with a very small imaginary part is by no means lost. It is of great importance to recognize such roots, as we thus see that a very slight change in the final coefficient, or in the conditions of the problem that led to the equation, would change these imaginary roots into equal, or nearly equal, real roots. The longer the recognition of their imaginary character is delayed, the nearer do the imaginary roots approach to being real roots, and the more reason is there to revise the data in accordance with which the equation was framed. Thus, in the cubic equation, $x^3 - \pi x - 2.143257 = 0$, if we employ the ordinary approximation $\pi = 3.1416$, we obtain three real roots, two nearly equal: but if we employ the closer approximation $\pi = 3.14159$, we obtain only one real root, the other two being imaginary with $\beta$ comparatively small.

**199.** In point of facility, the method of analysis here proposed presents obvious advantages over that of Sturm. In the latter method the difficulty of the analysis may be said to increase in geometrical ratio with the degree of the equation, and the calculation of the functions is equally operose, whether the roots, from their close proximity, are difficult of separation, or the reverse. In the method here recommended the labor of analysis may be said to increase only in arithmetical progression with the degree of the equation, and is, comparatively, but little affected by the magnitude of the coefficients; and the recognition of the character of the roots in a given interval is delayed only in proportion to the smallness of the quantity (real or imaginary) by which these roots differ. There is here no essential difference between the analysis and the solution of an equation, both being effected by one uniform process consisting of a series of transformations by Horner's Method, in which we first, by transforming over wider intervals, ascertain, by the rules established by Fourier's Theorem, in what intervals roots may lie; then, by transforming over narrower intervals, assisted by the trial divisors

(176), we either succeed in separating the roots, or find by the criterion established in (189) that certain roots are imaginary.

**200.** The calculation of imaginary roots may, theoretically, be effected by transforming, by methods given in Chap. XII, the equation in which they occur into others in which the real parts, $\alpha$ and $\beta$, occur separately as real roots, which may then be obtained by methods already given. Practically, however, when more than one, or at most two, pairs of imaginary roots occur in an equation, their calculation is very laborious. For some of the methods for determining such roots see Art. 212–215.

### EXERCISES.

Find the real roots of the following equations:

1. $x^4 + 2x^3 - 12x^2 - x + 2 = 0.$
2. $x^4 + 5x^3 - 15x^2 - 77x - 10 = 0.$
3. $x^4 - 46x^2 - 7x + 2 = 0.$
4. $x^4 + 2x^3 - 123x^2 + 478x - 154 = 0.$
5. $x^4 - 12x^3 - 90x^2 - 92x + 120 = 0.$
6. $x^4 - 16x^2 + 35x + 8 = 0.$
7. $x^4 - 7x^3 + 4x^2 + 20 = 0.$
8. $x^5 - 4x^4 - 16x^3 + 40x^2 + 80x + 32 = 0.$
9. $x^5 - 19x^3 + 97x^2 + 56x - 105 = 0.$
10. $x^5 - 32x^3 - 181x^2 + 5792 = 0.$
11. $x^5 - 37x^3 + x^2 - 30x + 5 = 0.$
12. $x^5 - 3663x^3 - 75856x^2 + 258962x + 792076 = 0.$
13. $x^5 - 6x^4 + x^3 + 13x^2 + 138x - 315 = 0.$
14. $x^5 - x^4 - 20x^3 + 43x^2 + 2x - 14 = 0.$
15. $x^5 + 3x^4 - 72x^3 + 4x^2 + 1020x - 1164 = 0.$
16. $x^5 - 152x^3 - 1874x^2 + 284848 = 0.$
17. $x^6 - 38x^4 - 40x^3 + 217x^2 + 160x - 215 = 0.$
18. $x^6 - 12x^5 + 9x^4 + 109x^3 - 34x^2 - 267x - 26 = 0.$
19. $x^6 - 3x^5 + 3x^4 - 78x^3 + 165x^2 - 165x + 1265 = 0.$
20. $x^6 + x^4 + 7x^2 - 6x + 6 = 0.$

# CHAPTER XI.

## CUBIC EQUATIONS.

**201.** A cubic equation, being of odd degree, has always a real root of sign contrary to that of the final term (11); the leading figure of a root of a cubic can thus be always ascertained by trying which is the least number that will cause the final term to change sign. Then, having calculated the root to which this number is an immediately superior limit, we can in various ways determine the remaining roots. This manner of proceeding may, however, under certain circumstances prove more laborious than is necessary, and the determination of the remaining roots is not so easily effected as in the method of procedure about to be explained, by which the tentative character of the usual method is avoided, and the roots obtained with a minimum of calculation.

**202.** As in Art. 122, let the roots $x_1$, $x_2$, $x_3$, of $x^3 + qx + r = 0$, be $2a$, $-(a+\beta)$, $(-a-\beta)$, in order of numerical value, and, as before, writing the coefficients in terms of the roots, we have,

$$x^3 - (3a^2 + \beta^2)x - (2a^2 - 2a\beta^2) = 0.$$

Hence $3a^2 + \beta^2 = -q$;

$\therefore \quad 4a^2 \quad = -q$, when $\beta = a$, *i. e.*, when one root is indefinitely small,

and $\quad 2a \quad = \sqrt{-q}$, the smallest possible value of $x_1$, the sign of $\sqrt{-q}$ being taken contrary to that of $r$.

Again, as $3a^2+\beta^2 = -q$,

$\therefore \quad 3a^2 \quad = -q$, when $\beta = 0$, *i. e.*, when there are two equal roots,

and $\quad 2a \quad = \sqrt{-\frac{4}{3}q}$, the greatest possible value of $x_1$, when the roots are all real.

Again, as $3\alpha^2+\beta^2 = -q$,

$\therefore \quad 3\alpha^2-\gamma^2 = -q$, when $\beta = \gamma\sqrt{-1}$, *i.e.*, when there are two imaginary roots,

and $\quad 2\alpha \quad > \sqrt{-\frac{4}{3}q}$, since $3\alpha^2 > -q$. In this last case, if $\gamma^2 =$ or $> 3\alpha^2$, then $q$ is zero or positive, and we see by inspection that there are imaginary roots.

Hence, if all the roots are real, the numerically greatest must lie between the limits $\sqrt{-q}$ and $\sqrt{-\frac{4}{3}q}$, with the sign of $-r$; also, since the superior limit $\sqrt{-\frac{4}{3}q}$ is equal to the inferior limit $\sqrt{-q}$ multiplied by $\sqrt{\frac{4}{3}}$, which equals $\sqrt{\frac{48}{36}}$, or $\frac{7}{6}$ nearly, the inferior limit differs from the superior by less than $\frac{1}{7}$ of the latter. Therefore, if we add one-third of itself to $-q$, the leading figure of the square root of the sum will be either the first figure of $x_1$, or may be a unit too great,* if the roots are all real.

**203.** Again, since

$$-r = 2\alpha^3 - 2\alpha\beta^2,$$

$$\begin{aligned}\therefore \quad \sqrt[3]{-4r} = \sqrt[3]{8\alpha^3-8\alpha\beta^2} &< 2\alpha \text{ when } \beta \text{ is real,}\\ &= 2\alpha \text{ when } \beta = 0,\\ &> 2\alpha \text{ when } \beta = \gamma\sqrt{-1}.\end{aligned}$$

Comparing these with the results of the preceding article, we find that $\sqrt{-\frac{4}{3}q} >, =,$ or $< \sqrt[3]{-4r}$, according as the roots are all real and unequal, there are two equal roots, or there are two imaginary roots, so that in every case $x_1$ lies between $\sqrt{-\frac{4}{3}q}$ and $\sqrt[3]{-4r}$, both having the sign of the latter. We are now prepared to state the following rule :

*Given an equation $x^3 + qx + r = 0$, where $q$ is negative, we determine by inspection the leading figures of $\sqrt{-\frac{4}{3}q}$ and $\sqrt[3]{-4r}$, taking both with the sign of the latter, then* (1), *if the first figure thus found is greater than the second, the roots are all real and unequal and the first figure is either the leading figure of $x_1$ or may be a unit too great;* (2), *if the first figure is less than the second, there are two imaginary*

* In comparatively rare cases the leading figure of $\sqrt{-\frac{4}{3}q}$ may be greater than the leading figure of $x_1$ by two units, as when $-q$ is close on a square number, and at the same time the equation has one root very small as compared with the others.

*roots, and the leading figure of* $x_1$ *is not greater than the leading figure of* $\sqrt{-4r}$; (3), if both figures are equal, we have in each the leading figure of $x_1$, but the remaining roots are doubtful. See foot of page 59.

Ex. 1. $$x^3 - 51x - 62 = 0.$$

Here, adding one-third of itself to 51 we have 68, and $62 \times 4 = 248$; we see by inspection that $\sqrt{68} = 8 +$ and $\sqrt[3]{288} = 6 +$, the roots are therefore all real and unequal, and the leading figure of $x_1$ is 8 or 7.

Ex. 2. $$x^3 - 65x + 316 = 0.$$

Here $\sqrt{-\frac{4}{3}q} = -9+$, and $\sqrt[3]{-4r} = -10+$, there are therefore two imaginary roots, and the leading figure of $x_1$ is $-10$, or $-9$.

Ex. 3. $$x^3 - 2778x - 56429 = 0.$$

Here $\sqrt{-\frac{4}{3}q} = 60 +$, and $\sqrt[3]{-4r} = 60 +$, the leading figure of $x_1$ is certainly 60; the remaining roots are doubtful. If it is desirable to ascertain their character without solving the equation, this may easily be effected by comparing $27r^2$ and $4q^3$.

**204.** We are in this way able to determine the leading figure of $x_1$ within very narrow limits. This root is also the most convenient for calculation, as there is no other of the same sign; it also differs from the others by $3\alpha \pm \beta$, that is, by at least its own value, so that the trial divisors will be found quite as effective in guiding us to the second and succeeding figures of the root as they are in the operation for the cube root, which, as before mentioned, is but the solution of a cubic of the form $x^3 + r = 0$.

**205.** When $x_1$ has been found, we may, if it is commensurable, depress the equation to a quadratic and thus find $x_2$, $x_3$. In few cases, however, will the root be commensurable; it will therefore be found most convenient, after approximating to $x_1$ to six or seven decimals, to calculate the remaining roots by one of the formulæ deduced as follows:

Since $3\alpha^2 + \beta^2 = -q$, $\therefore\ \beta = \pm\sqrt{-q-3\alpha^2}$, hence

$$-(\alpha \pm \beta) = -\frac{x_1}{2} \pm \sqrt{-q - \tfrac{3}{4}x_1^2}.$$

Or, since $2\alpha(\alpha^2 - \beta^2) = -r$, and $3\alpha^2 + \beta^2 = -q$, we obtain

$$4\beta^2 = -q + \frac{3r}{2\alpha}, \quad \therefore\ -(\alpha \pm \beta) = -\tfrac{1}{2}\left(x_1 \pm \sqrt{-q + \frac{3r}{x_1}}\right)$$
$$= x_2 \text{ or } x_3.$$

Ex. 1. $$x^3 - 7x - 7 = 0.$$

Here adding one-third of itself to 7, and multiplying 7 by 4, we obtain $9+$ and 28, the square root of the first and the cube root of the second both begin with 3, which must therefore be the leading figure of $x_1$. For comparison we place, on the following page, side by side, the approximation to $x_1$ of this equation, and the extraction of the cube root of 28.342421, which is the cube of $x_1$.

It will be observed that some modifications have been introduced into the process for the calculation of $x_1$, which is thus rendered strictly analogous to that usually employed for the extraction of the cube root. Thus, after completing the right-hand column by the addition of three ciphers,* we complete the second column by adding the square of the figure last found to the two rows of figures above, and then appending two ciphers; the left-hand column we complete by adding twice the figure last found, and append one cipher. The contraction is performed as usual, according to the rules in (170).

To find the roots $x_2$, $x_3$, we shall employ the second formula, thus,

$$x_2 \text{ and } x_3 = -\tfrac{1}{2}\left(3.0489173 \pm \sqrt{7 - \frac{21}{3.0489173}}\right)$$
$$= -\tfrac{1}{2}(.30489173 \pm \sqrt{.3351253})$$
$$= -1.692021.. \text{ and } -1.356896...$$

* Or by bringing down three decimal figures, if there are any, to the right of the integer, as in the cube root operation.

<table>
<tr><td>0</td><td>−7</td><td>−7</td><td>3.0489173</td></tr>
<tr><td>3</td><td>9</td><td>6</td><td></td></tr>
<tr><td>3</td><td>2</td><td>−1000000</td><td></td></tr>
<tr><td>6</td><td>9</td><td>814464</td><td></td></tr>
<tr><td>900</td><td>200000</td><td>−185536000</td><td></td></tr>
<tr><td>4</td><td>3616</td><td>166382592</td><td></td></tr>
<tr><td>904</td><td>203616</td><td>−19153408</td><td></td></tr>
<tr><td>8</td><td>16</td><td>18791228</td><td></td></tr>
<tr><td>9120</td><td>20724800</td><td>−362180</td><td></td></tr>
<tr><td>8</td><td>73024</td><td>208875</td><td></td></tr>
<tr><td>9128</td><td>20797824</td><td>−153305</td><td></td></tr>
<tr><td>16</td><td>64</td><td>146212</td><td></td></tr>
<tr><td>|91|44</td><td>2087091|2</td><td>−7093</td><td></td></tr>
<tr><td></td><td>823|0</td><td>6266</td><td></td></tr>
<tr><td></td><td>2087914|2</td><td>−827</td><td></td></tr>
<tr><td></td><td>823|</td><td></td><td></td></tr>
<tr><td></td><td>208873|7</td><td></td><td></td></tr>
<tr><td></td><td>|9</td><td></td><td></td></tr>
<tr><td></td><td>2088|7|4|6</td><td></td><td></td></tr>
</table>

<table>
<tr><td>0</td><td>0</td><td>−28.342421</td><td>3.0489173</td></tr>
<tr><td>3</td><td>9</td><td>27</td><td></td></tr>
<tr><td>3</td><td>9</td><td>−1342421</td><td></td></tr>
<tr><td>6</td><td>9</td><td>1094464</td><td></td></tr>
<tr><td>900</td><td>270000</td><td>−247957000</td><td></td></tr>
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<tr><td>904</td><td>273616</td><td>−25574408</td><td></td></tr>
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<tr><td>9120</td><td>27724800</td><td>−483180</td><td></td></tr>
<tr><td>8</td><td>73024</td><td>278873</td><td></td></tr>
<tr><td>9128</td><td>27797824</td><td>−204305</td><td></td></tr>
<tr><td>16</td><td>64</td><td>195212</td><td></td></tr>
<tr><td>|91|44</td><td>2787091|2</td><td>−9093</td><td></td></tr>
<tr><td></td><td>823|0</td><td>8366</td><td></td></tr>
<tr><td></td><td>2787914|2</td><td>−727</td><td></td></tr>
<tr><td></td><td>823|</td><td></td><td></td></tr>
<tr><td></td><td>278873|7</td><td></td><td></td></tr>
<tr><td></td><td>|9</td><td></td><td></td></tr>
<tr><td></td><td>2788|7|4|6</td><td></td><td></td></tr>
</table>

Ex. 2. $x^3 - 25x + 148 = 0.$

Here $25 \times \frac{4}{3} = 33+$, and $-148 \times 4 = -592$; the square root of the first is $-5+$, the cube root of the second is $-8+$; there are therefore imaginary roots, and Cardan's Formula could be employed to find $x_1$. It is obvious, however, that the compact process above exemplified will give the root with much less numerical labor. The leading figure of $x_1$ must lie between $-5$ and $-8$; upon trial, it is found to be 6, we then proceed as follows, first changing the sign of the final term, so as to make $x_1$ positive:

<table>
<tr><td>0</td><td>—25</td><td>—148</td><td>6.8312004</td></tr>
<tr><td>6</td><td>36</td><td>66</td><td></td></tr>
<tr><td>6</td><td>11</td><td>—82000</td><td></td></tr>
<tr><td>12</td><td>36</td><td>78432</td><td></td></tr>
<tr><td>180</td><td>8300</td><td>—3568000</td><td></td></tr>
<tr><td>8</td><td>1504</td><td>3429987</td><td></td></tr>
<tr><td>188</td><td>9804</td><td>—138013</td><td></td></tr>
<tr><td>16</td><td>64</td><td>114967</td><td></td></tr>
<tr><td>2040</td><td>1137200</td><td>—23046</td><td></td></tr>
<tr><td>3</td><td>6129</td><td>22998</td><td></td></tr>
<tr><td>2043</td><td>1143329</td><td>—48</td><td></td></tr>
<tr><td>6</td><td>9</td><td>45</td><td></td></tr>
<tr><td>|20|49</td><td>114946|7</td><td>—3</td><td></td></tr>
<tr><td></td><td>20|5</td><td></td><td></td></tr>
<tr><td></td><td>114967|2</td><td></td><td></td></tr>
<tr><td></td><td>20|</td><td></td><td></td></tr>
<tr><td></td><td>11498|7</td><td></td><td></td></tr>
<tr><td></td><td>|4</td><td></td><td></td></tr>
<tr><td></td><td>11|4|9|9|1</td><td></td><td></td></tr>
</table>

Changing the sign of the root thus found, we have $x_1 = -6.8312004$. The imaginary roots, if desired, may be obtained by one of the formulæ above given.

Ex. 3. $$7x^3 - 443x - 1396 = 0.$$

Here we must divide the coefficients by 7 and proceed as in the previous examples; thus $\frac{443}{7} \times \frac{4}{3} = 84+$, and $\frac{1396}{7} \times 4 = 797+$; as the square root of 84 and the cube root of 797 both begin with 9, this must be the leading figure of $x_1$.

Ex. 4. $$x^3 + 56x - 181 = 0.$$

Here we see at once that two roots are imaginary, but can apply the rules for limits given in (75, 76). By the latter $\frac{181}{57} + 1$ is a superior limit; the root proves to be $2.82\ldots$

Ex. 5. $$x^3 - 11x^2 - 102x - 181 = 0.$$

In order to make the second coefficient divisible by 3, we multiply the roots by 3 (59); we thus obtain,

$$y^3 - 33y^2 - 918y - 4887 = 0.$$

We proceed to diminish the roots by 11, and obtain,

$$z^3 + 0 - 1281z - 17647 = 0.$$

As $\sqrt{1281 \times \frac{4}{3}}$ and $\sqrt[3]{17647 \times 4}$ both begin with 40, we proceed to calculate $z_1$ with 40 as leading figure and obtain 41.3279469. For the remaining roots we have,

$$z_2,\ z_3 = -\tfrac{1}{2}\left(41.3279469 \pm \sqrt{1281 - \frac{52491}{41.3279469}}\right)$$
$$= -20.6960396,\ -20.6319072.$$

Adding 11 to each of these roots, and dividing by 3, we obtain the roots of the proposed equation,

$$x_1 = 17.4426489,\quad x_2 = -3.2106357,\quad x_3 = -3.2320132.$$

EXERCISES.

Solve the following equations :

1. $x^3 - 3x - 1 = 0.$
2. $x^3 - 63x - 9 = 0.$
3. $x^3 - 51x - 62 = 0.$
4. $x^3 - 51x - 68 = 0.$
5. $x^3 - 7x + 5 = 0.$
6. $x^3 - 6x + 3 = 0.$

7. $x^3 - 37x - 151 = 0.$
8. $x^3 - 9x + 5 = 0.$
9. $x^3 - 78x - 150 = 0.$
10. $x^3 - 87x + 12 = 0.$
11. $x^3 - 43x - 4532 = 0.$
12. $x^3 - 88x - 251 = 0.$
13. $x^3 - 8x + 35 = 0.$
14. $x^3 - 93x - 15 = 0.$
15. $x^3 - 56x + 7 = 0.$
16. $x^3 - 29x + 54 = 0.$
17. $x^3 - 25x - 148 = 0.$
18. $x^3 - 31x + 45 = 0.$
19. $x^3 - 15x - 2 = 0.$
20. $x^3 - 10x - 13 = 0.$
21. $x^3 - 71x - 85 = 0.$
22. $x^3 - 17x - 23 = 0.$
23. $x^3 - 23x + 35 = 0.$
24. $x^3 - 75x - 8 = 0.$
25. $x^3 - 43x - 6 = 0.$
26. $x^3 - 93x - 51 = 0.$
27. $x^3 - 88x - 251 = 0.$
28. $x^3 - 57x - 1205 = 0.$
29. $x^3 - 65x + 987 = 0.$
30. $x^3 - 108x - 431 = 0.$
31. $x^3 - 132x - 443 = 0.$
32. $x^3 - 12x + 17 = 0.$
33. $x^3 - 172x + 342 = 0.$
34. $x^3 - 149x - 530 = 0.$
35. $x^3 - 184x - 269 = 0.$
36. $x^3 - 456x + 789 = 0.$
37. $x^3 - 73x - 268 = 0.$
38. $x^3 - 345x - 678 = 0.$
39. $x^3 - 234x - 567 = 0.$
40. $x^2 - 91x - 331 = 0.$
41. $x^3 - 123x + 649 = 0.$
42. $x^3 - 123x - 456 = 0.$
43. $x^3 - 678x - 901 = 0.$
44. $x^3 - 567x - 890 = 0.$
45. $x^3 - 234x - 3634 = 0.$
46. $x^3 - 368x + 6531 = 0.$
47. $x^3 - 547x - 4924 = 0.$
48. $x^3 - 890x - 423 = 0.$
49. $x^3 - 496x - 33721 = 0.$
50. $x^3 - 432x + 3557 = 0.$

51. $x^3 - 20x^2 + 27x - 58 = 0.$
52. $x^3 - 59x^2 + 1135x - 7137 = 0.$
53. $x^3 + 26x^2 + 262x + 808 = 0.$
54. $x^3 - 49x^2 + 658x - 1379 = 0.$
55. $4x^3 - 180x^2 + 1896x - 457 = 0.$
56. $27x^3 + 27x^2 - 180x + 127 = 0.$
57. $512x^3 + 192x^2 - 10728x + 9409 = 0.$
58. $5x^3 + 124.2x^2 - 338.065x - 18606.379 = 0.$
59. $12x^3 - 120x^2 + 326x - 127 = 0.$
60. $4x^3 - 240x^2 + 3996x - 14937 = 0.$

# CHAPTER XII.

## SYMMETRICAL FUNCTIONS OF THE ROOTS.

**206.** A symmetrical function of the roots of an equation is an expression in which the roots are similarly involved, so that the function is not altered if any of the roots be interchanged. Thus the coefficients of an equation are symmetrical functions of the roots, since (45) in the equation

$$x^n + C_{n-1}x^{n-1} + C_{n-2}x^{n-2} + \dots C_1 x + C_0 = 0$$

$$\begin{aligned} -C_{n-1} &= a_1 + a_2 + a_3 + \dots a_n, \\ C_{n-2} &= a_1 a_2 + a_1 a_3 + a_2 a_3 + \dots a_{n-1} a_n, \\ -C_{n-3} &= a_1 a_2 a_3 + a_1 a_2 a_4 + \dots a_{n-2} a_{n-1} a_n, \end{aligned}$$

and so on. These are symmetrical functions; since however we may interchange the roots, the function is not altered. We propose in the present chapter to show that by means of the above elementary functions, given by the coefficients of the equation, we can obtain any rational symmetrical function of the roots of an equation in terms of its coefficients.

**207.** A function,

$$a_1^m + a_2^m + a_3^m + \dots a_n^m,$$

in which each term involves only one of the roots, is said to be of the *first order.*

A function,

$$a_1^m a_2^p + a_1^m a_3^p + a_2^m a_3^p + \dots a_{n-1}^m a_n^p \dots,$$

which contains every permutation of the roots taken two at a time, with $m$ as the exponent of the first and $p$ of the second, is said to be of the *second order,* and is usually denoted by $\Sigma a_1^m a_2^p$, as being the sum of all the terms that can be formed like $a_1^m a_2^p$.

A function,

$$a_1^m a_2^p a_3^q + a_1^m a_3^p a_4^q + a_2^m a_3^p a_4^q + \dots,$$

which contains every permutation of the roots taken three at a time, with $m$ as the exponent of the first, $p$ of the second, and $q$ as the third, is said to be of the *third order*, and is denoted by $\Sigma a_1^m a_2^p a_3^q$. For functions of the fourth and higher orders symbols of similar form may be employed.

The function of the first order, which is the sum of the $m^{th}$ powers of the roots, might in like manner be denoted by $\Sigma a_1^m$, but is usually denoted simply by $S_m$, as in the following proposition containing Newton's Theorem for obtaining the functions of the first order.

**208.** *To express the sum of the $m^{th}$ powers of the roots in terms of the coefficients and inferior powers.*

Let $a_1, a_2, a_3, \dots a_n$ denote the roots of the equation

$$f(x) = x^n + C_{n-1}x^{n-1} + C_{n-2}x^{n-2} + \dots C_1 x + C_0 = 0,$$

also

$$\begin{aligned} S_1 &= a_1 + a_2 + a_3 + \dots\dots\dots\dots a_n, \\ S_2 &= a_1^2 + a_2^2 + a_3^2 + \dots\dots\dots\dots a_n^2, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ S_m &= a_1^m + a_2^m + a_3^m + \dots\dots\dots\dots a_n^m. \end{aligned}$$

By Art. 92 we have

$$f_1(x) = \frac{f(x)}{x - a_1} + \frac{f(x)}{x - a_2} + \dots \frac{f(x)}{x - a_n}.$$

Each of the divisions here indicated can be performed exactly by Art. 19, the quotient being the equation depressed by division by one of its binomial factors; thus we have,

$$\begin{aligned} \frac{f(x)}{x - a_1} &= x^{n-1} + (a_1 + C_{n-1})x^{n-2} + (a_1^2 + C_{n-1}a_1 + C_{n-2})x^{n-3} + \dots \\ &\quad + (a_1^m + C_{n-1}a_1^{m-1} + C_{n-2}a^{m-2} + \dots C_{n-m})x^{n-m-1} + \dots \\ \frac{f(x)}{x - a_2} &= x^{n-1} + (a_2 + C_{n-1})x^{n-2} + (a_2^2 + C_{n-1}a_2 + C_{n-2})x^{n-3} + \dots \\ &\quad + (a_2^m + C_{n-1}a_2^{m-1} + C_{n-2}x^{m-2} + \dots C_{n-m})x^{n-m-1} + \dots \end{aligned}$$

with similar results for $\frac{f(x)}{x - a_3}$, $\frac{f(x)}{x - a_4}$, &c.

Hence, by the addition of these $n$ quotients, we have

$$f_1(x) = nx^{n-1} + (S_1 + nC_{n-1})x^{n-2} + (S_2 + C_{n-1}S_1 + nC_{n-2})x^{n-3} + \dots$$
$$+ (S_m + C_{n-1}S_{m-1} + C_{n-2}S_{m-2} + \dots nC_{n-m})x^{n-m-1} + \dots$$

But, by Art. 7, we have also,

$$f_1(x) = nx^{n-1} + (n-1)C_{n-1}x^{n-2} + (n-2)C_{n-2}x^{n-3} + \dots\dots$$
$$+ (n-m)C_{n-m}x^{n-m-1} + \dots\dots$$

Equating the coefficients of corresponding terms in these identical expressions, we have,

$$S_1 + nC_{n-1} = (n-1)C_{n-1}, \text{ or } S_1 + C_{n-1} = 0.$$
$$S_2 + C_{n-1}S_1 + nC_{n-2} = (n-2)C_{n-2}, \text{ or } S_2 + C_{n-1}S_1 + 2C_{n-2} = 0.$$
$$S_3 + C_{n-1}S_2 + C_{n-2}S_1 + nC_{n-3} = (n-3)C_{n-3}, \text{ or}$$
$$S_3 + C_{n-1}S_2 + C_{n-2}S_1 + 3C_{n-3} = 0.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$S_m + C_{n-1}S_{m-1} + C_{n-2}S_{m-2} + \dots C_{n-m-1}S_1 + nC_{n-m} = (n-m)C_{n-m}, \text{ or}$$
$$S_m + C_{n-1}S_{m-1} + C_{n-2}S_{m-2} + \dots C_{n-m-1}S_1 + mC_{n-m} = 0.$$

This last formula gives $S_m$, the sum of the $m^{th}$ powers of the roots, in terms of the coefficients and inferior powers of the roots. Thus having $S_1$, we can find $S_2$, and then $S_3$, and so on, till we reach $S_m$, where $m$ is less than $n$.

The process may be extended so as to obtain $S_m$, where $m$ is not less than $n$, as follows :

Multiplying the given equation by $x^{m-n}$, we have

$$x^m + C_{n-1}x^{m-1} + C_{n-2}x^{m-2} + \dots\dots C_0x^{m-n} = 0.$$

In this substituting in succession $a_1, a_2, a_3, \dots\dots a_n$, for $x$, and adding the results, we obtain

$$S_m + C_{n-1}S_{m-1} + C_{n-2}S_{m-2} + \dots\dots C_0S_{m-n}.$$

Hence, making $m = n$, $n+1$, $n+2$, &c., in succession, we find, observing that $S_0 = a_1^0 + a_2^0 + \dots\dots a_n^0 = n$,

$$S_n + C_{n-1}S_{n-1} + C_{n-2}S_{n-2} + \dots\dots nC_0 = 0,$$
$$S_{n+1} + C_{n-1}S_n + C_{n-2}S_{n-1} + \dots\dots C_0S_1 = 0,$$

and so on till we obtain $S_m$, where $m$ may be of any magnitude.

Ex. To find $S_6$, the sum of the sixth powers of the roots of

$$x^4 - 2x^3 - 5x^2 + 7x - 3 = 0.$$

Here $-C_3 = 2,\ -C_2 = 5,\ -C_1 = -7,\ -C_0 = 3$.
$S_1 = -C_3 = 2$.
$S_2 = -C_3 S_1 - 2C_2 = 4 + 10 = 14$.
$S_3 = -C_3 S_2 - C_2 S_1 - 3C_1 = 28 + 10 - 21 = 17$.
$S_4 = -C_3 S_3 - C_2 S_2 - C_1 S_1 - 4C_0 = 34 + 70 - 14 + 12 = 102$.
$S_5 = -C_3 S_4 - C_2 S_3 - C_1 S_2 - C_0 S_1 = 204 + 85 - 98 + 6 = 197$.
$S_6 = -C_3 S_5 - C_2 S_4 - C_1 S_3 - C_0 S_2 = 394 + 510 - 119 + 42 = 827$.

Thus the sum of the sixth powers of the roots is 827; the sums of higher powers can be found by continuing the process.

To find the sums of the negative powers of the roots we put $\frac{1}{y}$ for $x$, that is, transform the equation into the equation involving the reciprocals of the roots (64), and apply the formulæ as above.

**209.** A very convenient process for finding the sums of the powers of the roots may be deduced as follows:

$$\text{Since } f_1(x) = \frac{f(x)}{x - a_1} + \frac{f(x)}{x - a_2} + \frac{f(x)}{x - a_3} + \dots.$$

$$\therefore \quad \frac{xf_1(x)}{f(x)} = \frac{x}{x - a_1} + \frac{x}{x - a_2} + \frac{x}{x - a_3} + \dots.$$

$$= \left(1 - \frac{a_1}{x}\right)^{-1} + \left(1 - \frac{a_2}{x}\right)^{-1} + \left(1 - \frac{a_3}{x}\right)^{-1} + \dots$$

$$= n + S_1 x^{-1} + S_2 x^{-2} + S_3 x^{-3} + \dots.$$

That is, if we multiply $f_1(x)$ by $x$, and divide by $f(x)$ the coefficients of the quotient will be in their order $S_0$, $S_1$, $S_2$, $\dots . S_m$. Thus, in the example of the preceding article, if we divide $4x^4 - 6x^3 - 10x^2 + 7x$ by $x^4 - 2x^3 - 5x^2 + 7x - 3$, we obtain in succession, as the coefficients of the quotient, 4, 2, 14, 17, 102, &c.

**210.** From the equations in (208), expressing the sums of the powers of the roots in terms of the coefficients, we can obtain expressions for the coefficients in terms of $S_1$, $S_2$, $S_3$, &c.; thus,

$$
\begin{aligned}
C_{n-1} &= -S_1, \\
C_{n-2} &= -\tfrac{1}{2}(S_2 + C_{n-1}S_1), \\
C_{n-3} &= -\tfrac{1}{3}(S_3 + C_{n-1}S_2 + C_{n-2}S_1), \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
C_{n-r} &= -\frac{1}{r}(S_r + C_{n-1}S_{r-1} + \ldots . C_{n-r-2}S_1).
\end{aligned}
$$

**211.** *Any rational symmetrical function of the roots of an equation can be expressed in terms of the coefficients and functions of lower order.*

First, to find the value of $\Sigma a_1^m a_2^p$, the function of the second order.

Since
$$
\begin{aligned}
S_m &= a_1^m + a_2^m + a_3^m + a_4^m + \ldots . \\
S_p &= a_1^p + a_2^p + a_3^p + a_4^p + \ldots .
\end{aligned}
$$

by multiplication we obtain,

$$
\begin{aligned}
S_m S_p = {} & a_1^{m+p} + a_2^{m+p} + a_3^{m+p} + a_4^{m+p} + \ldots . \\
& + a_1^m a_2^p + a_1^m a_3^p + a_1^m a_4^p + a_2^m a_3^p + \ldots .
\end{aligned}
$$

that is,

$$S_m S_p = S_{m+p} + \Sigma a_1^m a_2^p, \text{ or } \Sigma a_1^m a_2^p = S_m S_p - S_{m+p}. \quad [1].$$

Next, to find the value of $\Sigma a_1^m a_2^p a_3^q$, the function of the third order, we multiply together the equations

$$
\begin{aligned}
\Sigma a_1^m a_2^p &= a_1^m a_2^p + a_1^m a_3^p + a_1^m a_4^p + a_2^m a_3^p + \ldots . \\
S_q &= a_1^q \quad + a_2^q \quad + a_3^q \quad + a_4^q \quad + \ldots .
\end{aligned}
$$

and obtain a result consisting of three partial products :

(1) the sum of the products of the form
$$a_1^{m+q} a_2^p = \Sigma a_1^{m+q} a_2^p,$$

(2) the sum of the products of the form
$$a_1^{p+q} a_2^m = \Sigma a_1^{p+q} a_2^m,$$

(3) the sum of the products of the form
$$a_1^m a_2^p a_3^q = \Sigma a_1^m a_2^p a_3^q;$$

thus,

$$S_q \times \Sigma a_1^m a_2^p = \Sigma a_1^{m+q} a_2^p + \Sigma a_1^{p+q} a_2^m + \Sigma a_1^m a_2^p a_3^q.$$

Substituting for $\Sigma a_1^m a_2^p$, $\Sigma a_1^{m+q} a_2^p$, and $\Sigma a_1^{p+q} a_2^m$ their values obtained by formula [1], we have

$$\Sigma a_1^m a_2^p a_3^q = S_m S_p S_q - S_{m+q} S_p - S_{p+q} S_m + 2S_{m+p+q}. \quad [2].$$

By proceeding in a similar manner we can obtain formulæ for functions of the fourth and higher orders expressed in terms of the sums of the powers of the roots; and as these (208) can be expressed by integral functions of the coefficients, any rational symmetrical function $\Sigma a_1^m a_2^p \ldots\ldots a_m^r$ can be expressed by integral functions of the coefficients.

These are called the elementary symmetrical functions; and, as it is by the combination of these that every complex, rational, and symmetrical function is formed, it follows that the value of every rational symmetrical function of the roots can be expressed by the coefficients of the equation.

**212.** The formulæ obtained above, where the exponents $m$, $p$, $q$, &c., were assumed to be unequal, must be modified if we suppose any of the exponents to be equal. Thus, if $m = p$ in the formula

$$\Sigma a_1^m a_2^p = S_m S_p - S_{m+p},$$

since then $a_1^m a_2^p = a_1^p a_2^m$, the terms in $\Sigma a_1^m a_2^p$ become equal, two and two, and $\Sigma a_1^m a_2^p$ becomes $2\Sigma(a_1 a_2)^m$, therefore, in this case,

$$\Sigma a_1^m a_2^m = \tfrac{1}{2}(S_m^2 - S_{2m}).$$

Similarly, if, in $\Sigma a_1^m a_2^p a_3^q$, $m = p$, we have

$$\Sigma(a_1 a_2)^m a_3^q = \tfrac{1}{2}(S_m^2 S_q - S_{2m} S_q - 2S_{m+q} S_m + 2S_{2m+q});$$

and if $m = p = q$, $\Sigma a_1^m a_2^m a_3^m$ becomes $2 \cdot 3\Sigma(a_1 a_2 a_3)^m$; or

$$\Sigma(a_1 a_2 a_3)^m = \tfrac{1}{6}(S_m^3 - 3S_{2m} S_m + 2S_{3m}).$$

**213.** By means of the theory of symmetrical functions we are able to transform an equation into another the roots of which shall be given functions of those of the proposed equation. The following transformation, having, at one time, been proposed as a mean of separating the roots of an equation, is of some interest.

**214.** *To transform an equation into another whose roots are the squares of the differences of the roots of the proposed equation.*

Let $a_1, a_2, a_3, \ldots. a_n$, be the roots of the proposed equation, then the roots of the required transformed equation will be

$$(a_1 - a_2)^2, \ (a_1 - a_3)^2, \ldots.. (a_2 - a_3)^2, \text{\&c.}$$

and will be in number $\frac{1}{2}n(n-1)$, the possible number of different combinations of $n$ things taken two at a time. The degree of the transformed equation will therefore be $\frac{1}{2}n(n-1) = m$ suppose.

Let $$y^m + q_1 y^{m-1} + q_2 y^{m-2} + \ldots. q_m = 0$$

be the transformed equation, and $s_1, s_2, \ldots. s_r$ denote the sums of the first, second, .... $r^{th}$ powers of its roots. When we have determined the values of these sums of the powers we can obtain the coefficients, since (210) $q_1 = -s_1$, $q_2 = -\frac{1}{2}(s_2 + q_1 s_1)$, &c.; it remains therefore to find a general expression for any sum as $s_r$.

Let $S_1, S_2, \ldots. S_m$, denote, as before, the sums of the powers of the roots of the proposed equation, then

$$(x - a_1)^{2r} + (x - a_2)^{2r} + (x - a_3)^{2r} + \ldots. (x - a_n)^{2r}$$
$$= nx^{2r} - 2rS_1 x^{2r-1} + \frac{2r(2r-1)}{1 \cdot 2} S_2 x^{2r-2} + \ldots. S_{2r}.$$

In the first member of the above equation, when we put $a_1$ for $x$, we obtain the sum of the $r^{th}$ powers of the squares of the differences in which $a_1$ comes first; when we put $a_2$ for $x$, we obtain the sum of the $r^{th}$ powers of the squares of the differences in which $a_2$ comes first, and so on; if therefore we put $a_1, a_2, a_3, \ldots. a_n$, in succession for $x$, and add all the results, we obtain

$$2s_r = nS_{2r} - 2rS_1 S_{2r-1} + \frac{2r(2r-1)}{1 \cdot 2} S_2 S_{2r-2} - \ldots. nS_{2r}.$$

The terms to the right hand that are equidistant from the beginning and end are equal; collecting and dividing by 2, we have

$$s_r = nS_{2r} - 2rS_1 S_{2r-1} + \frac{2r(2r-1)}{1 \cdot 2} S_2 S_{2r-2} + \ldots.$$
$$+ \frac{1}{2}(-1)^r \frac{2r(2r-1) \ldots. (r+1)}{\lfloor r} S_r^2.$$

To obtain the required transformation, we accordingly first find $S_1, S_2, S_3, \ldots.$ by (208) from the coefficients of the proposed equation; we then, by means of the formula just given, find in succession $s_1, s_2, \ldots. s_m$; and, finally, by means of the formulas $q_1 = -s_1$, $q_2 = -\frac{1}{2}(s_2+q_1 s_1)$, &c., obtain the coefficients of the required equation.

The impracticable nature of a method of analysis requiring, as a mere preliminary to the separation of the roots of an equation of the sixth degree, the calculation of fifteen coefficients through these three stages, is self-evident.

**215.** The theory of symmetrical functions may also be applied to the elimination of one of the unknown quantities between two simultaneous equations. See Art. 227.

**216.** The sums of the powers of the roots may be advantageously employed in certain cases to obtain an approximation to the roots of an equation.*

Let $a_1, a_2, a_3, \ldots.$ denote the roots of an equation in descending order of magnitude, then we have

$$\frac{S_{m+1}}{S^m} = \frac{a_1^{m+1} + a_2^{m+1} + a_3^{m+1} + \ldots.}{a_1^m + a_2^m + a_3^m + \ldots.}$$

$$= a_1 \cdot \frac{1 + \left(\frac{a_2}{a_1}\right)^{m+1} + \left(\frac{a_3}{a_1}\right)^{m+1} + \ldots.}{1 + \left(\frac{a_2}{a_1}\right)^m + \left(\frac{a_3}{a_1}\right)^m + \ldots.}$$

Now the fractions $\left(\frac{a_2}{a_1}\right)^{m+1}$, $\left(\frac{a_2}{a_1}\right)^m$, &c., may be made as small as we like by taking $m$ large enough. $\frac{S_{m+1}}{S_m}$ is therefore an approximation to the greatest real root $a_1$, provided that it be greater than the modulus of any imaginary pair, the approximation becoming closer as $m$ increases.

**217.** But if there be a pair of imaginary roots whose modulus is greater than the greatest real root, we cannot ob-

* This was first suggested by Newton, and was further developed by Lagrange. The subject will be found very fully treated in Murphy's Equations.

tain an approximation to that root as above. For (Trig.) these imaginary roots may be put under the form $\mu(\cos\theta \pm \sin\theta\sqrt{-1})$, where $\mu$ is the modulus; then

$$\frac{S_{m+1}}{S_m} = \mu \cdot \frac{2\cos(m+1)\theta + \left(\frac{a_2}{\mu}\right)^{m+1} + \left(\frac{a_3}{\mu}\right)^{m+1} + \dots}{2\cos m\theta \quad + \left(\frac{a_2}{\mu}\right)^{m} + \left(\frac{a_3}{\mu}\right)^{m} + \dots}$$

$S_{m+1} \div S_m$ therefore approximates to $\mu \cdot \dfrac{\cos(m+1)\theta}{\cos m\theta}$, which may have any value.

**218.** Again, if the two greatest roots, $a_1$, $a_2$, are real,

$$S_m = (a_1^m + a_2^m)\left(1 + \frac{a_3^m}{a_1^m + a_2^m} + \frac{a_4^m}{a_1^m + a_2^m} + \dots\right);$$

if $a_1$ and $a_2$ are imaginary, then

$$S_m = \mu^m\left[2\cos\theta + \left(\frac{a_3}{\mu}\right)^m + \left(\frac{a_4}{\mu}\right)^m + \right].$$

In either case, therefore, we have

$$\begin{aligned} S_m &= a_1^m \;\; + a_2^m, \text{ nearly, when } m \text{ is taken large enough.} \\ S_{m+1} &= a_1^{m+1} + a_2^{m+1}, \text{ nearly,} \\ S_{m+2} &= a_1^{m+2} + a_2^{m+2}, \text{ nearly,} \end{aligned}$$

each equation being more nearly true than the preceding one;

$$\therefore \quad S_m \; S_{m+2} - S_{m+1}^2 = (a_1 a_2)^m \;(a_1 - a_2)^2 = u_m,$$
$$S_{m+1} S_{m+3} - S_{m+2}^2 = (a_1 a_2)^{m+1}(a_1 - a_2)^2 = u_{m+1};$$
$$\therefore \quad \frac{u_m}{u_{m+1}} = a_1 a_2, \text{ approximatively.}$$

That is, if from every three terms of the series $S_1$, $S_2$, $S_3$, &c., another series $\Sigma u_m$ be formed by subtracting the square of the means from the product of the extremes, the quotients obtained by dividing each term of this new series by the term that precedes it, approximate more and more nearly to the product of the two greatest roots. If the two greatest roots are real, then the greater being known by (216), the second becomes known by this process. If they are imaginary, we can proceed to find their sum as follows.

**219.** As in the preceding case, we can find from the values of $S_m$, $S_{m+1}$, $S_{m+2}$, &c., that

$$S_m S_{m+3} - S_{m+1} S_{m+2} = a_1^m a_2^m (a_1 + a_2)(a_1 - a_2)^2 \text{ nearly}, = v_m \text{ say}.$$

Dividing this by $u_m$, we obtain $a_1 + a_2$ nearly; that is, if from every four terms of the series $S_1, S_2, S_3, \ldots.$ another series $\Sigma v_m$ be formed by subtracting the product of the means from the product of the extremes, then the quotients obtained by dividing each term of this series by the corresponding term of the series $\Sigma u_m$ approach more and more nearly to the sum of the two greatest roots.

The sum and the product of two imaginary roots having been found in this way, each can then be determined by a quadratic.

Ex. $$x^4 - x^3 + 4x^2 + x - 4 = 0.$$

$\Sigma S_m = 1, -7, -14, 29, 96, -34, -503, -347, 2083, 3838, -6159.$
$\Sigma u_m = -63, -399, -2185, -10202, -49444, -241211, -1168158.$
$\Sigma v_m = -69, -266, -2308, -11323, -50414, -245363, -1207713.$

The first series being divergent shows that the roots $a_1$, $a_2$ are imaginary. From the second series we obtain $a_1 a_2 = \dfrac{1168156}{241211} = 4.84$; from the second and third we have $a_1 + a_2 = \dfrac{1207713}{1168158} = 1.03$; the roots $a_1$, $a_2$ are, therefore, $\frac{1}{2}(1.03 \pm 4.3\sqrt{-1})$.

For the purpose of calculating real roots this method is obviously too laborious if we aim at accuracy to several places of decimals; but in some cases this appears to be the most convenient process yet proposed for approximating to the real and imaginary parts of impossible roots.

**220.** If an equation of any degree has only two imaginary roots, they can easily be determined, if required, in the following manner, which may be advantageously employed to determine the remaining roots of any equation when all but two have been calculated.

Let $\kappa_1$ and $\kappa_m$ denote respectively the sum and product of the ascertained values of all the roots but two, and $t_1$ and $t_2$

the sum and product of the remaining two; then (45) $t_1 = -(C_{n-1} + \kappa_1)$, and $t_2 = (-1)^n C_0 \div \kappa_m$; the two roots, therefore, can be obtained from the formula, $x = \frac{1}{2}(t_1 \pm \sqrt{t_1^2 - 4t_2})$.

Ex. 1. In the equation already given,

$$x^4 - x^3 + 4x^2 + x - 4 = 0,$$

we see at once (189), since $1 - 4 \times 2$ is negative, that there are two imaginary roots, and the final sign being negative, the other two are real, one of each sign. By Horner's Method the values of these roots, .892232.. and −.923262.., are calculated accurately to six places of decimals with less labor than is required to obtain the first series $\Sigma S_m$. The sum of these roots is −.03103, therefore $t_1 = 1.03103$; the product of the above roots is −.823765; dividing the final term −4 by this, we obtain $t_2 = 4.855762..$; hence we obtain for the roots $\frac{1}{2}(1.03103.. \pm 4.284857..\sqrt{-1})$. One advantage in obtaining the last two roots in this manner is that we are enabled to perform much of the calculation by logarithms.

Ex. 2. In the equation,

$$x^5 + 173x^4 + 2356x^3 + 10468x^2 - 14101x + 4183 = 0,$$

there is a pair of imaginary roots. The sum of the real roots is found to be −157.438702.., hence $t_1 = (-173 + 157.438702) = -15.561297...$ The product of the real roots is −49.9555..; dividing −4183 by this, we obtain $t_2 = 83.73454..$; hence we find the roots to be $-7.780648 \pm \sqrt{-23.19607}$.

**221.** If all the roots of an equation of the fourth degree are imaginary, we can always obtain their values by means of Descartes' reducing cubic (129). If, then, an equation of any degree has only four imaginary roots, we can, after obtaining the real roots, depress it to a biquadratic, and then obtain the imaginary roots. Unless, therefore, an equation has as many as six imaginary roots, we can depress the equation to one capable of algebraical solution, and thus obtain the values of the imaginary roots with comparative facility. But if an equation of the sixth degree, or one that has been depressed to that degree, has all its roots imaginary, the method of Art. 219

appears to afford the readiest solution. We can first, by that method, find the two greatest of these roots; then, by taking the coefficients in reverse order, and repeating the process, we obtain the reciprocals of the two smallest roots; finally, having thus determined four roots, we can easily obtain the sum and product of the remaining pair, and thus determine them also.

**222.** Theoretically the determination of the values of imaginary roots may be effected as follows. If in the proposed equation $f(x) = 0$, we substitute $a + \beta\sqrt{-1}$ for $x$, we obtain (26) a result of the form $P + Q\sqrt{-1} = 0$; hence we obtain two equations, $P = 0$, and $Q = 0$, each involving $a$ and $\beta$. As will be shown in the next chapter, we can from these equations always determine the corresponding values of $a$ and $\beta$. But, if $f(x)$ be of the sixth degree, this method requires the formation and part solution of an auxiliary equation of the fifteenth degree, while, if $f(x)$ is of the eighth degree, the auxiliary equation rises to the 28th degree.

EXERCISES.

Find the values of $S_2, S_3, \ldots S_6$, in the following equations:

1. $x^3 - 5x^2 + 6x - 1 = 0$.
2. $x^3 + 12x + 8 = 0$.
3. $x^4 + 2x^3 - 3x^2 - 4x + 1 = 0$.
4. $x^4 - 3x^3 - 7x + 5 = 0$.
5. $x^5 + 6x^3 + 5x^2 - x + 2 = 0$.
6. $x^5 - 6x^2 - 10 = 0$.

Find the remaining two roots of the following equations, the sums $\kappa_1$, and products $\kappa_{n-2}$, of the other roots being given:

1. $x^4 - 8x^3 + 16x^2 - x - 2 = 0$;
 $\kappa_1 = 7.864328$, $\kappa_2 = 13.41444$.
2. $x^4 + 8x^3 + 51x^2 + 30x + 30 = 0$;
 $\kappa_1 = -12.830412$, $\kappa_2 = 5.142960$.
3. $x^4 - 19x^2 - 20x + 5 = 0$;
 $\kappa_1 = 1.173254$, $\kappa_2 = 17.335048$.
4. $x^5 - x^4 - 11x^3 - 3x^2 + 23x + 13 = 0$;
 $\kappa_1 = 4.678823$, $\kappa_3 = -3.678823$.
5. $x^5 - 3x^4 - 10x^3 - 11x^2 + 121x + 70 = 0$;
 $\kappa_1 = 7.073375$, $\kappa_3 = -8.146750$.
6. $x^6 - 19x^4 - 22x^3 + 84x^2 + 179x + 85 = 0$;
 $\kappa_1 = 4.027525$, $\kappa_4 = -20.366725$.

# CHAPTER XIII.

## ELIMINATION.

**223.** Two equations involving each two unknown quantities $x$ and $y$ may be denoted by $F(x, y) = 0$, and $f(x, y) = 0$; their solution consists in the determination of the systems of values of $x$ and $y$ which cause both equations to vanish simultaneously. In certain cases the solution is readily effected. Thus, if from one of the equations we are able to obtain an expression for the value of one of the unknown quantities, as $x$, in terms of the other $y$, by the substitution of this value for $x$ in the other equation we obtain an equation involving $y$ only, the roots of which can be determined by methods already given, and these values of $y$ substituted in the expression for $x$ in terms of $y$, will give the corresponding values of $x$.

Ex.
$$\begin{aligned} x^3 - (y+2)x^2 + (y^2 - 2y)x - 12 &= 0, \\ x^2 - yx \qquad + 2 &= 0. \end{aligned}$$

From the second equation we obtain $y = x + \dfrac{2}{x}$. This value of $y$ substituted in the first equation gives us

$$x^4 - 2x^3 + 4x^2 - 4x - 8 = 0.$$

Of this, one value is $x = 2$, and for this value of $x$, $y = 3$, which values cause both equations to vanish simultaneously. The other values of $x$, being determined, would in like manner give corresponding values of $y$.

**224.** Also, if the first members of the equations can be readily resolved into factors, the solution of the equations may be made to depend upon that of equations of inferior degrees. If, for example, we find $F(x, y) = UU'U'' = 0$, and $f(x, y) = VV' = 0$, then the systems of values that satisfy the pro-

posed equations may be obtained by solving the simultaneous equations $U = 0$ and $V = 0$; $U = 0$ and $V' = 0$; $U' = 0$ and $V = 0$; $U' = 0$ and $V' = 0$; $U'' = 0$ and $V = 0$; $U'' = 0$ and $V' = 0$.

Let $F(x, y) = x^3y - (y^2-2y)x^2 + (y^3+2y)x - y^4 - 2y^2 = 0$,
$f(x, y) = x^3 - (y+1)x^2 - (2-y)x + 2 = 0.$

These may be resolved as follows,

$$F(x, y) = (x^2 - xy + y^2)(xy - y^2 - 2) = 0,$$
$$f(x, y) = (x^2 - xy - 2)(x + y - 1) = 0.$$

The systems of values that satisfy the proposed equations can therefore be obtained by solving the equations

$$\left.\begin{array}{l} x^2 - xy + y^2 = 0 \\ x^2 - xy - 2 = 0 \end{array}\right\}, \quad \left.\begin{array}{l} x^2 - xy - y^2 = 0 \\ x + y - 1 = 0 \end{array}\right\},$$

$$\left.\begin{array}{l} xy - y^2 - 2 = 0 \\ x^2 - xy - 2 = 0 \end{array}\right\}, \quad \left.\begin{array}{l} xy - y^2 - 2 = 0 \\ x + y - 1 = 0 \end{array}\right\}.$$

**225.** But if one of the factors of $F(x, y)$ is identical with one of the factors of $f(x, y)$; if, for example, $U$ and $V$ are identical, then, obviously, any values of $x$ and $y$ that make $U$ vanish will cause both the proposed equations to vanish simultaneously. If $U$ involves both $x$ and $y$, we can assign any value we please to one of these unknowns, and determine the corresponding value, or values, of the other that will cause $U$ to vanish, and thus obtain as many solutions as we please. If $U$ involves only one of the unknown quantities, we can satisfy the equations $F(x, y) = 0$, and $f(x, y) = 0$ by giving to that unknown quantity any value that causes $U$ to vanish, and to the other any value we please. In order, therefore, that the proposed equations be determinate, that is, be satisfied by a limited number of corresponding values of $x$ and $y$, they must not have a common divisor involving $x$ or $y$.

**226.** To eliminate between two equations, each involving the same two unknown quantities, is to deduce an equation, involving only one of these unknown quantities, the solution of which will furnish all the values of that quantity, which, taken with the corresponding values of the other, satisfy the

proposed equations. The equation thus deduced is called the final equation, and its roots are called suitable values.

**227.** *To eliminate one of the unknown quantities between two equations involving two unknown quantities by means of symmetrical functions.*

Let
$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots . p_n = 0, \qquad [1]$$
$$q_0x^m + q_1x^{m-1} + q_2x^{m-2} + \dots . q_m = 0, \qquad [2]$$

be the proposed equations, in which the coefficients $p_0$, $p_1$, $p_2$, .... $q_0$, $q_1$, $q_2$, .... are rational integral functions of $y$.

Assume that we can solve equation [1] in respect to $x$, in terms of $y$, and that these values are $a$, $b$, $c$, &c. By substituting these values in equation [2], we obtain $n$ equations involving only $y$, namely:

$$\left.\begin{aligned} q_0a^m + q_1a^{m-1} + q_2a^{m-2} + \dots . q_m &= 0 \\ q_0b^m + q_1b^{m-1} + q_2b^{m-2} + \dots . q_m &= 0 \\ q_0c^m + q_1c^{m-1} + q_2c^{m-2} + \dots . q_m &= 0 \end{aligned}\right\} \quad [3].$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

In general the solution of [1] cannot be effected; but by multiplying together all the above $n$ equations, we obtain a final equation which has for roots all the suitable values of $y$. For this equation will vanish for any of the values of $y$ that makes any of its factors vanish, and for no others; and any of these is a suitable value. For suppose the first of the equations [3] vanishes for $y = \beta$, and that when $\beta$ is put for $y$ in the function $a$, the value is $\alpha$; then $x = \alpha$, $y = \beta$ will satisfy the two proposed equations. Now in the final equation, which is the product of all the equations [3], the factors only change places when we interchange any of the quantities $a$, $b$, $c$, ...., thus the product is a symmetrical function of these quantities, which may be expressed in terms of the coefficients $p_0$, $p_1$, $p_2$,.... of equation [1], and thus we can obtain the final equation in $y$ which contains all the suitable values, and no other.

**228.** Though the preceding method of elimination has the merit of furnishing a final equation with all the suitable values,

and no others, the calculations required are so tedious that the method is not generally available. Upon it is based, however, the following theorem:

**229.** *The degree of the final equation resulting from the elimination of one of the unknown quantities between two equations of the $m^{th}$ and $n^{th}$ degrees respectively, will not exceed $mn$.*

Let
$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots p_n = 0,$$
$$q_0x^m + q_1x^{m-1} + q_2x^{m-2} + \dots q_m = 0,$$

be the proposed equations in which the coefficients are functions of $y$. Here it is supposed that the sum of the exponents of $x$ and $y$ does not exceed $n$ in any term of the first equation, or $m$ in the second, so that a coefficient $p_r$ or $q_r$ is a function of $y$ not higher than the $r^{th}$ degree.

Suppose $x$ to be eliminated by the method of Art. 228; then the first member of the final equation in $y$ consists of a series of terms, each of which is the product of $n$ factors, and is of the form $q_r a^{m-r} \times q_s b^{m-s} \times q_t c^{m-t} \times \dots$. Since (227) the series of terms forms a symmetrical function of $a, b, c, \dots$, the aggregate of the terms, with exponents as above, is,

$$q_r q_s q_t \dots \Sigma a^{m-r} b^{m-s} c^{m-t} \dots$$

Now the degree of $y$ in $q_r q_s q_t \dots$ is not higher than $r + s + t + \dots$, and it will be shown that the degree of $y$ in $\Sigma a^{m-r} b^{m-s} c^{m-t} \dots$ cannot exceed $m-r+m-s+m-t+\dots$, so that the degree of the whole product is not greater than $mn$. For from the formulæ in Art. 208 we see that $S_p$, for example, does not contain any power of $y$ greater than $y^p$; and from the formulæ in Art. 211 a function $\Sigma a^h b^k c^l \dots$ will contain powers and products of $S_1, S_2, \dots S_{h+k+l+\dots}$, in each term the sum of the letters subscript to $S$ being $h+k+l+\dots$; the degree of $y$ in the function $\Sigma a^{m-r} b^{m-s} c^{m-t} \dots$ cannot therefore exceed $m+r+m-s+m-t+\dots$; hence no power of $y$ in the final equation can be higher than $y^{mn}$.

**230.** Although the degree of $y$ in the final equation cannot exceed $mn$, it may in certain cases be less. By an extension of the process to any number of equations we are led to the general theorem discovered by Bezout, namely:

If between any number of equations involving the same number of unknown quantities we eliminate all but one, the degree of the final equation will not exceed the product of the degrees of the original equations.

**231.** The elimination of one of the unknown quantities between two equations is most conveniently effected by the following method, based upon the operation for obtaining the greatest common measure of two algebraical quantities.

*To determine the systems of values that will satisfy two equations involving two unknown quantities.*

Let $F(x, y) = 0$ and $f(x, y) = 0$ be two equations, which we shall suppose have no common factor and thus admit of a limited number of solutions. Suppose that $x = a$ and $y = \beta$ are values that satisfy these equations, then $F(a, y) = 0$ and $f(a, y) = 0$ are both satisfied by $y = \beta$. $F(a, y)$ and $f(a, y)$ must therefore have a common measure which, equated to zero, will have for a root a value of $y$ which, conjointly with $x = a$, will satisfy the proposed equations. Having therefore arranged both equations according to descending powers of $x$, we proceed by the operation for the G. C. M. till we arrive at a remainder independent of $x$, say $\phi(y)$.

Now if no factors, functions of $y$, have been introduced or suppressed in the operation, so as to avoid having $y$ in a denominator, $\phi(y) = 0$ will comprise all the suitable values of $y$, and no others. For unless $\phi(y) = 0$, $F(a, y)$, and $f(a, y)$ have no common measure and therefore do not vanish simultaneously. But if it has been necessary to introduce factors, functions of $y$, then some of the roots of $\phi(y) = 0$ may not be suitable values, having been introduced with these factors; and if factors have been suppressed, there may be suitable values of $y$ not found among the roots of $\phi(y) = 0$.

**232.** A process by which the suitable values of $y$ may be determined is furnished in the following method due to M. M. Labatie and Sarrus.

Let $A = 0$ and $B = 0$ be the two simultaneous equations, of which neither has a factor a function of $y$ only, and $B$ is not of higher dimensions in $x$ than $A$. Let $c$ denote the

factor by which $A$ must be multiplied to make it divisible by $B$; let $q$ be the quotient, and $rR$ the remainder, where $r$ is a function of $y$ only. Let $c_1$ denote the factor by which $B$ must be multiplied to make it divisible by $R$, let $q_1$ be the quotient, and $r_1 R_1$ the remainder, where $r_1$ is a function of $y$ only. Suppose that, proceeding in this way, at the third division we arrive at a remainder $r_2$ independent of $x$. Thus we have the identities,

$$\left.\begin{aligned} c\,A &= q\,B + r\,R, \\ c_1 B &= q_1 R + r_1 R_1, \\ c_2 R &= q_2 R_1 + r_2. \end{aligned}\right\} \qquad [1].$$

Let $d$ be the G. C. M. of $c$ and $r$, $d_1$ the G. C. M. of $\frac{c\,c_1}{d}$ and $r_1$, $d_2$ the G. C. M. of $\frac{c\,c_1\,c_2}{d\,d_1}$ and $r_2$. We have now to show that the solution of the equations $A = 0$, $B = 0$, will be obtained by solving the equations:

$$\left.\begin{aligned} \frac{r}{d} &= 0 \\ B &= 0 \end{aligned}\right\}, \quad \left.\begin{aligned} \frac{r_1}{d_1} &= 0 \\ R &= 0 \end{aligned}\right\}, \quad \left.\begin{aligned} \frac{r_2}{d_2} &= 0 \\ R_1 &= 0 \end{aligned}\right\}. \qquad [2].$$

I. *All the solutions obtained from this system of equations satisfy the equations* $A = 0$, $B = 0$.

Dividing both members of the first identity [1] by $d$, we have

$$\frac{c}{d}A = \frac{q}{d}B + \frac{r}{d}R. \qquad [3].$$

Since $d$ is the G. C. M. of $c$ and $r$, $\frac{c}{d}$ and $\frac{r}{d}$ are integral functions of $y$, therefore also $\frac{q}{d}B$ is an integral function; but, by hypothesis, $B$ has no factor which is a function of $y$ only, therefore $d$ must divide $q$.

From [3] we see that the values of $x$ and $y$ that satisfy the equations $\frac{r}{d} = 0$ and $B = 0$ make $\frac{c}{d}A$ also vanish; but $\frac{r}{d}$ and $\frac{c}{d}$ have no common factor, therefore these values make $A$ vanish. Hence all the solutions of $\frac{r}{d} = 0$ and $B = 0$, satisfy $A = 0$, $B = 0$.

Again multiplying both members of the identity [3] by $c_1$, and substituting for $c_1B$ its equivalent from the second identity in [1], we have

$$\frac{cc_1}{d}A = \frac{c_1r + qq_1}{d}R + \frac{q}{d}r_1R_1.$$

The expression $\frac{c_1r + qq_1}{d}$ is integral, since $r$ and $q$ are divisible by $d$; the expression is also divisible by $d_1$, for $d_1$ divides $\frac{cc_1}{d}$ and $r_1$, and does not divide $R$. Divide by $d_1$, then putting $M$ for $\frac{q}{d}$ and $M_1$ for $\frac{c_1r + qq_1}{dd_1}$, we have

$$\frac{cc_1}{dd_1}A = M_1R + \frac{r_1}{d_1}MR. \qquad [4].$$

Multiplying both members of the second identity [1] by $\frac{c}{d}$, we have

$$\frac{cc_1}{d}B = \frac{cq_1}{d}R + \frac{c}{d}r_1R_1.$$

Since $d_1$ divides $\frac{cc_1}{d}$ and $r_1$, it will divide $\frac{cq_1}{d}R$; but $R$ is not divisible by $d_1$, therefore $\frac{cq_1}{d}$ must be so. Dividing by $d_1$ and putting $N$ for $\frac{c}{d}$ and $N_1$ for $\frac{cq_1}{dd_1}$, we have

$$\frac{cc_1}{dd_1}B = N_1R + \frac{r_1}{d_1}NR_1. \qquad [5].$$

The identities [4] and [5] show that all the values of $x$ and $y$ that make $\frac{r_1}{d_1}$ and $R$ vanish make $\frac{cc_1}{dd_1}A$ and $\frac{cc_1}{dd_1}B$ vanish, but $\frac{cc_1}{dd_1}$ and $\frac{r_1}{d_1}$ have no common factor; therefore all the solutions of the equations $\frac{r_1}{d_1} = 0$ and $R = 0$ satisfy the equations $A = 0$, $B = 0$.

In the same way, if we multiply both members of the identies [4] and [5] by $c_2$, and substitute for $c_2R$ its equivalent from the third identity [1], we obtain

$$\frac{cc_1c_2}{dd_1d_2}A = M_2R_1 + \frac{r_2}{d_2}M_1, \qquad [6].$$

$$\frac{cc_1c_2}{dd_1d_2}B = N_2R_1 + \frac{r_2}{d_2}N_1, \qquad [7].$$

where $M_2$ and $N_2$ are integral functions of $x$ and $y$. From these identities we see that all the solutions of the equations $\frac{r_2}{d_2} = 0$ and $R_1 = 0$, satisfy the equations $A = 0$, $B = 0$.

II. *All the values of $x$ and $y$ that satisfy the equations $A = 0$, $B = 0$ are included among the solutions obtained from the systems of equations* [2].

The identity [3] may be written

$$NA - MB = \frac{r}{d}B. \qquad [8].$$

Multiply [4] by $B$, [5] by $A$, and subtract, then

$$(M_1 B - N_1 A)R + (MB - NA)\frac{r_1}{d_1}R_1 = 0,$$

therefore, by [8],

$$(M_1 B - N_1 A)R - \frac{r\,r_1}{d\,d_1}RR_1 = 0,$$

therefore $\qquad M_1 B - N_1 A \quad = \frac{r\,r_1}{d\,d_1}R_1. \qquad [9].$

In the same way, from [6] and [7], we may deduce

$$M_2 B - N_2 A \quad = \frac{r\,r_1\,r_2}{d\,d_1\,d_2}R_2 \qquad [10].$$

This last identity shows that all the values of $x$ and $y$ that make $A$ and $B$ vanish make $\frac{r\,r_1\,r_2}{d\,d_1\,d_2}$ vanish. Hence the equations

$$\frac{r}{d} = 0, \ \frac{r_1}{d_1} = 0, \ \frac{r_2}{d_2} = 0,$$

supply all the suitable values of $y$.

Suppose that $x = \alpha$, $y = \beta$ are values that satisfy the equations $A = 0$, $B = 0$, then

(1). If $\beta$ is a root of the equation $\frac{r}{d} = 0$, then the values $x = \alpha$, $y = \beta$ evidently satisfy the equations $\frac{r}{d} = 0$ and $B = 0$.

(2). If $\beta$ is not a root of $\frac{r}{d} = 0$, but is of $\frac{r_1}{d_1} = 0$, then, since $\frac{r}{d}$ does not vanish when $y = \beta$, it follows from [8] that

the values $x = \alpha$, $y = \beta$, make $R$ vanish; thus they satisfy $\dfrac{r_1}{d_1} = 0$ and $R = 0$.

(3). If $\beta$ is not a root of the equation $\dfrac{r}{d} = 0$, nor of $\dfrac{r_1}{d_1} = 0$, but is of the equation $\dfrac{r_2}{d_2} = 0$, then, since $\dfrac{r\,r_1}{d\,d_1}$ does not vanish when $y = \beta$, it follows from [9] that the values $x = \alpha$, $y = \beta$ make $R_1$ vanish; thus they satisfy the equations $\dfrac{r_2}{d_2} = 0$ and $R_1 = 0$.

The equation $\dfrac{r\,r_1\,r_2}{d\,d_1\,d_2} = 0$, which furnishes all the suitable values of $y$, may be called the *final equation* in $y$.

**233.** If, instead of a final remainder in $y$, we arrive at a remainder zero, this shows that $A$ and $B$ have the preceding remainder, say $R_1$, as a common factor. Then, as we saw in Art. 225, the equations $A = 0$ and $B = 0$ will be satisfied by an unlimited number of values of $x$ and $y$ derived from the equation $R_1 = 0$. By dividing by this common measure, we obtain $\dfrac{A}{R_1}$ and $\dfrac{B}{R_1}$, which have no common factor, then $\dfrac{A}{R_1} = 0$ and $\dfrac{B}{R_1} = 0$ will be satisfied for a limited number of values of $x$ and $y$, which may be found as above.

If the final remainder is a mere number, say $l$, then the equation $l = 0$ being absurd shows that the proposed equations are incompatible with each other.

Ex. 1. $x^3 + 3yx^2 + (3y^2 - y + 1)x + y^3 - y^2 + 2y = 0.$
$x^2 + 2yx + y^2 - y = 0.$

Here the first remainder is $x + 2y$, so that $r = 1$; the second remainder is $y^2 - y$, which is independent of $x$. All the solutions are furnished by $\dfrac{r_1}{d_1} = 0$ and $R = 0$, that is, by the equations $y^2 - y = 0$, and $x + 2y = 0$.

Ex. 2. $x^3 + 2yx^2 + 2y(y-2)x + y^2 - 4 = 0.$
$x^2 + 2yx + 2y^2 - 5y + 2 = 0.$

The first remainder is $(y-2)(x+y+2)$; so that $r = y-2$,

$R = x + y + 2$; the second remainder is $y^2 - 5y + 6$, which is independent of $x$. All the solutions are furnished by $\frac{r}{d} = 0$ and $B = 0$, that is, by the equations $y - 2 = 0$ and $x^2 + 2xy + 2y^2 - 5y + 2 = 0$; and by $\frac{r_1}{d_1} = 0$ and $R = 0$, that is, by $x^2 - 5y + 6 = 0$ and $x + y + 2 = 0$. The final equation in $y$ is $(y-2)(y^2 - 5y + 6) = 0$.

Ex. 3. $$x^3+3yx^2-3x^2+3y^2x-6yx-x-y^3-3y^2-y+3 = 0.$$
$$x^3-3yx^2+3x^2+3y^2x-6yx-x-y^3+3y^2+y-3 = 0.$$

The first remainder is $2(y-1)(3x^2 + y^2 - 2y - 3)$; the second remainder is $8(y^2 - 2y)x$; the third, which is independent of $x$, is $y^2 - 2y - 3$. The solutions are furnished by the systems of equations,

$y-1=0$ and $x^3-3yx^2+3x^2+3y^2x-6yx-x-y^3+3y^2+y-3=0$;
$y^2 - 2y = 0$ and $3x^2 + y^2 - 2y - 3 = 0$;
$y^2 - 2y - 3 = 0$ and $x = 0$.

The final equation in $y$ is $(y-1)(y^2-2y)(y^2-2y-3) = 0$.

Ex. 4. $$yx^3 - (y^3 - 3y - 1)x + y = 0.$$
$$x^2 - y^2 + 3 = 0.$$

The first remainder is $x + y$; the second is $3$; the proposed equations are therefore incompatible.

### EXERCISES.

Solve the following equations :

1. $\left.\begin{array}{l} x^3 - x^2 + (2y^2 - 34)x - 2y^2 + 34 = 0 \\ x^2 - (y+7)x - 2y^2 + 4y = 0 \end{array}\right\}.$

2. $\left.\begin{array}{l} (y-1)x^2 + yx + y^2 - 2y = 0 \\ (y-1)x + y = 0 \end{array}\right\}.$

3. $\left.\begin{array}{l} x^2 + (8y - 13)x + y^2 - 7y + 12 = 0 \\ x^2 - (4y + 1)x + y^2 + 5y = 0 \end{array}\right\}.$

4. $\left.\begin{array}{l} (y-1)x^3 + y(y+1)x^2 + (3y^2 + y - 2)x + 2y = 0 \\ (y-1)x^2 + y(y+1)x + 3y^2 - 1 = 0 \end{array}\right\}.$

5. $\left.\begin{array}{l} (y-2)x^2 - 2x + 5y - 2 = 0 \\ yx^2 - 5x + 4y = 0 \end{array}\right\}.$

# ANSWERS.

## PAGE 4.

1. $f(0) = -13,\ f(1) = -7,\ f(2) = 71,\ f(3) = 299.$
2. $f(0) = 18,\ f(1) = 26,\ f(2) = 546,\ f(3) = 7416.$
3. $f(0) = -16,\ f(1) = -51,\ f(2) = -268,\ f(3) = -511.$
4. $f(0) = 4,\ f(1) = -7,\ f(2) = -528,\ f(3) = -7331.$
5. $f_r(x) = n(n-1)\ldots.(n-r+1)C_n x^{n-r} + (n-1)(n-2)\ldots.$
$(n-r)C_{n-1}x^{n-r-1} + \ldots.\underline{|r}C_r.$

## PAGE 15.

1. $f(3) = 184.$
2. $f(4) = -198.$
3. $f(5) = -1646.$
4. $f(-2) = 1449.$
5. $f(11) = 85814289.$

## PAGE 27.

1. $f(x) = (x-3)(x-3)(x-2)(x+7).$
2. $f(x) = (x-8)(x-5)(x+4)(x+2)(x+1).$
3. $f(x) = x^3 - 93x + 308 = 0.$
4. $f(x) = x^4 - 12x^3 + 49x^2 - 78x + 40 = 0.$
5. $f(x) = x^4 + 4x^3 - 79x^2 - 106x + 840 = 0.$
6. $f(x) = x^5 - 25x^4 + 220x^3 - 832x^2 + 1480x - 1600 = 0.$

## PAGE 28.

1. $x = 3 - \sqrt{-3},\quad -3 \pm \sqrt{12}.$
2. $x = -5 - \sqrt{-1},\quad \frac{1}{2}(5 \pm \sqrt{29}).$
3. $x = \frac{1}{2}(7 + \sqrt{-11}),\quad \frac{1}{12}(5 \pm \sqrt{73}).$
4. $x = \frac{1}{2}(1 - \sqrt{-3}),\quad -\frac{1}{5}(1 \pm \sqrt{-14}).$
5. $x = 4 - \sqrt{7},\quad -2,\quad -6.$

## PAGE 33.

1. $x = 3$.
2. $x = 5$.
3. $x = 4$.
4. $x = 2, 5$.
5. $x = 6$.

## PAGE 37.

1. $x^6 - 7x^4 + 3x + 5 = 0$.
2. $x^5 + 7x^3 - x - 2 = 0$.
3. $7y^{39} + 3y^{36} - 5y^{32} - 7 = 0$.
4. $2y^{32} + \frac{1}{2}y^{21} + 3 = 0$.
5. $y^{12} + y^6 + y^5 - y^4 - 2 = 0$.
6. $9x^{10} - x^9 - x^6 - 6x^5 + 1 = 0$.
7. $(1 - x)(1 + x)^3 = 0$.

## PAGE 39.

1. $y^3 - 3y^2 - 55y - 500 = 0$.
2. $y^3 - 54y^2 + 1470 = 0$.
3. $y^3 - 2357y^2 + 367000y - 745000 = 0$.
4. $y^4 + 2625y^2 - 154350y + 22509375 = 0$.

## PAGE 41.

6. The roots occur in pairs differing only in sign, thus $f(-x) = 0$ is identical with $f(x) = 0$.

## PAGE 42.

1. $11y^3 - 20y^2 + 5y - 3 = 0$.
2. $15y^4 + 72y^3 + 54y^2 - 7 = 0$.
3. $23y^5 - 7y^4 - 32y^3 + 17y - 1 = 0$.

## PAGE 44.

1. $x^8 - 145x^6 - 22x^4 - 156x^2 + 25 = 0$.
2. $25x^8 - 170x^6 + 479x^4 - 1175x^2 + 361 = 0$.
3. $x^{10} - 54x^8 + 657x^6 - 6156x^4 + 5796x^2 + 625 = 0$.

## PAGE 47.

1. $y^4 + 5y^3 - 2y^2 + 817y + 4050 = 0$.
2. $y^4 + 84y^3 + 332y^2 + 573y + 327 = 0$.
3. $11y^4 + 147y^3 + 708y^2 + 1480y + 1191 = 0$.
4. $3y^5 + 47y^4 + 313y^3 + 1068y^2 + 1780y + 1043 = 0$.
5. $8y^5 - 200y^4 + 2000y^3 - 9922y^2 + 24113y - 23055 = 0$.

## PAGE 48.

1. $y^3 - 33y - 71 = 0.$
2. $y^3 - 275y - 1692 = 0.$
3. $y^3 + 186y + 1807 = 0.$
4. $y^4 - 174y^2 - 1261y - 2549 = 0.$
5. $y^4 - 726y^2 - 6616y + 6045 = 0.$

## PAGE 53.

1. Superior limit 2, inferior limit $-15$.
2. Superior limit 4, inferior limit $-6$.
3. Superior limit 5, no negative root.
4. Superior limit 4, inferior limit $-5$.

## PAGE 64.

1. $f(x) = (x-2)^2(x-3) = 0.$
2. $f(x) = (x-\frac{3}{2})^2(x+6) = 0.$
3. $f(x) = (x-5)^2(x^2+10x+3) = 0.$
4. $f(x) = (x-\frac{7}{2})^2(2x^2-x-1) = 0.$
5. $f(x) = (x^2-2x-1)^2(x+5) = 0.$
6. $f(x) = (x^2+3x+1)^2(x^2-6x+12) = 0.$

## PAGE 67.

The reduced equations are:

1. $3y^2 - 7y + 25 = 0.$
2. $5y^2 + 8x - 66 = 0.$
3. $y^2 + 13y - 40 = 0.$
4. $2y^2 - 11y - 2 = 0.$
5. $y^2 - 23y - 1 = 0.$

## PAGE 74.

1. $x = \sqrt[3]{5}, \quad -\frac{1}{2}(1 \pm \sqrt{-3})\sqrt[3]{5}.$
2. $x = \pm\frac{1}{2}(\sqrt{2} \pm \sqrt{-2})\sqrt{2}.$
3. $x = \sqrt[5]{7}, \quad \frac{1}{4}\left(\sqrt{5} - 1 \pm \sqrt{-10 - 2\sqrt{5}}\right)\sqrt[5]{7},$
   $-\frac{1}{4}\left(\sqrt{5} + 1 \pm \sqrt{-10 + 2\sqrt{5}}\right)\sqrt[5]{7}.$
4. $x = \pm\sqrt[3]{2}, \quad \frac{1}{2}(1 \pm \sqrt{-3})\sqrt[3]{2}, \quad -\frac{1}{2}(1 \pm \sqrt{-3})\sqrt[3]{2}$
5. $x = \frac{1}{4}\left(\sqrt{5} - 1 \pm \sqrt{-10 - 2\sqrt{5}}\right),$
   $-\frac{1}{4}\left(\sqrt{5} + 1 \pm \sqrt{-10 + 2\sqrt{5}}\right).$
6. Multiply the roots in (5) by $\pm\sqrt[5]{3}$.

## PAGE 74.

7. The roots are all the products, with changed signs, of the roots of $x^5 - 1 = 0$ and $x^3 - 1 = 0$.
8. Multiply the roots in (7) by $-\sqrt[15]{20}$.

The depressed equations are:

1. $y^3 + y^2 - 2y - 1 = 0.$
2. $y^4 - y^3 - 3y^2 + 2y - 1 = 0.$
3. $y^5 - y^4 - 4y^3 + 3y^2 + 3y - 1 = 0.$
4. $z^4 + z^2 + 1 = 0.$

## PAGE 82.

The real roots are:

1. $x = -.32748\ldots$
2. $x = 4.107243\ldots$
3. $x = .896922\ldots$
4. $x = -.696549\ldots$
5. $x = 2.576222\ldots$
6. $x = -2.896025\ldots$

## PAGE 86.

1. $x = 5, \quad -2, \quad -\frac{1}{2}(3 \pm \sqrt{21}).$
2. $x = \frac{1}{2}(1 \pm \sqrt{-11}, \quad -\frac{1}{2}(1 \pm \sqrt{-3}).$
3. $x = 2, \quad -4, \quad 1 \pm 2\sqrt{-1}.$
4. $x = \frac{1}{2}(7 \pm \sqrt{37}, \quad \frac{1}{2}(5 \pm \sqrt{17}).$
5. $x = \frac{1}{10}(3 \pm \sqrt{69}), \quad -\frac{1}{2}(5 \pm \sqrt{-3}).$

## PAGE 95.

The depressed equations are:

1. $x^2 + x = \pm(2x + 23).$
2. $x^2 - \frac{17}{2}x = \pm\left(\frac{13}{2}x + 15\right).$
3. $x^2 - 4x + 2 = \pm 5.$
4. $x^2 - 10x + \frac{17}{2} = \pm\frac{23}{2}.$
5. $x^2 - \frac{7}{2}x + 5 = \pm\frac{13}{2}x.$
6. $x^2 - \frac{3}{2}x = \pm\left(\frac{3}{2}x - 31\right).$
7. $x^3 + 12x^2 - x - \frac{5}{2} = \pm\left(22x + \frac{5}{2}\right).$
8. $x^3 + 3x^2 = \pm(4x + 5).$

## PAGE 105.

1. [4, 5], two imaginary roots.
2. [6, 7], two imaginary roots.
3. [7, 8], two imaginary roots.
4. [$-.6$, $-.5$], [$-.5$, $-.4$], [2, 3], [4, 5].
5. Roots all imaginary.
6. [0, 1], four imaginary roots.

## PAGE 125.

1. $x = .6458252, .6458290, -52.6458252, -100.6458290.$
2. $x = .4142852, .4142843, -72.4142852, -142.4142843.$
3. $x = .7321401, .7321410, -114.7321401, -170.7321410.$
4. $x = .8285606, .8285659, -74.8285606, -144.8285659.$
5. $x = .4499705, .4499761, -84.4499705, -102.4499761.$
6. $x = .2360680, .2360663, -4.2360680,$ two imag. roots.
7. $x = .1414284, .1414246, -14.1414284,$ two imag. roots.
8. $x = .2396769, .2396118, -54.2396769,$ two imag. roots.
9. $x = .4422711, .4422496, -90{,}4422711,$ two imag. roots.
10. $x = .5612971, .5612977, -158{,}5612971,$ two imag. roots.

## PAGE 146.

1. $x = 2.618034, .381966, -4.561552, -.438447.$
2. $x = 5, 3.938004, -.123492, -3.804512.$
3. $x = 6.854102, .145898, -.298438, -6.701562.$
4. $x = 5.645751, 5.643651, .354249, -13.643651.$
5. $x = 17.440307, .732051, -2.732051, -3.440307.$
6. $x = -.208712, -4.791288,$ two imag. roots.
7. $x = 6.282681, 1.883579,$ two imag. roots.
8. $x = 4.82843, 3.75877, -.69461, -.82843, -3.06418.$
9. $x = .700263, -2.41742, -14.58258,$ two imag. roots.
10. $x = 5.656854, 5.656653, -5.656854,$ two imag. roots.
11. $x = 4.132933, .162278, -6.162278,$ two imag. roots.
12. $x = 68.63515, 4.63519, -2, -30.63519, -40.63515.$
13. $x = 3, 3, 4.073375,$ two imaginary roots.
14. $x = 2.87939, 2.87298, -.53209, -.65271, -4.87298.$
15. $x = 4.73265, 4.73205, 1.267949, -4.38548, -9.34717.$
16. $x = 12.328868, 12.328828, -12.328828,$ two imag. roots.

## PAGE 146.

17. $x = 6.18658$, $2.16601$, $.78281$, $-1.47848$, $-2.94883$, $-4.6081$.
18. $x = 10.09902$, $3.67882$, $2$, $-.90098$, two imag. roots.
19. $x = 3.80295$, $3.80204$, four imag. roots.
20. $x =$ No real roots.

## PAGE 152.

1. $x = 1.879385$, $-1.53209$.*
2. $x = 8.007741$, $-7.864837$.
3. $x = 7.685514$, $-6.431126$.
4. $x = 7.73265$, $-6.347169$.
5. $x = 2.166011$, $-2.948828$.
6. $x = 2.145102$, $-2.669079$.
7. $x = 7.549859$, two imaginary roots.
8. $x = 2.669443$, $-3.246542$.
9. $x = 9.670143$, $-7.639755$.
10. $x = 9.257633$, $-9.395601$.
11. $x = 17.414059$, two imaginary roots.
12. $x = 10.570921$, $-7.332814$.
13. $x = -4.073375$, two imaginary roots.
14. $x = 9.723306$, $-9.562042$.
15. $x = 7.420014$, $-7.545049$.
16. $x = 3.886699$, $-6.146939$.
17. $x = 6.8312004$, two imaginary roots.
18. $x = 4.608100$, $-6.186583$.
19. $x = 3.938004$, $-3.804512$.
20. $x = 3.678823$, two imaginary roots.
21. $x = 8.970896$, $-7.747852$.
22. $x = 4.681165$, $-3.09226$.
23. $x = 3.668800$, $-5.426743$.
24. $x = 8.713102$, $-8.606420$.
25. $x = 6.626123$, $-6.486525$.
26. $x = 9.906962$, $-9.356784$.
27. $x = 10.570917$, $-7.332828$.
28. $x = 12.412788$, two imaginary roots.
29. $x = -12.105179$, two imaginary roots.

* The remaining root is easily found by subtraction.

## PAGE 152.

30. $x = 11.996912, \quad -6.231475.$
31. $x = 12.897578, \quad -9.139393.$
32. $x = -4.027525$, two imaginary roots.
33. $x = 11.976841, \quad -14.014402.$
34. $x = 13.699870, \quad -9.719621.$
35. $x = 14.243786, \quad -12.764226.$
36. $x = 20.429925, \quad -22.171735.$
37. $x = 9.991179$, two imaginary roots.
38. $x = 19.488225, \quad -17.500218.$
39. $x = 16.388155, \quad -13.902291.$
40. $x = 11.003675$, two imaginary roots.
41. $x = -13.131062$, two imaginary roots.
42. $x = 12.615331, \quad -8.215575.$
43. $x = 26.679053, \quad -25.346654.$
44. $x = 24.560824, \quad -22.984299.$
45. $x = 20.318709$. two imaginary roots.
46. $x = -25.070094$, two imaginary roots.
47. $x = 27.006096, \quad -13.558683.$
48. $x = 30.067726, \quad -29.592326.$
49. $x = 37.388585$, two imaginary roots.
50. $x = -24.077597$, two imaginary roots.
51. $x = 18.7234001$, two imaginary roots.
52. $x = 25.190234, \quad 18.470444.$
53. $x = -5.312233$, two imaginary roots.
54. $x = 23.213112, \quad 23.229537.$
55. $x = 28.521277, \quad 16.231939.$
56. $x = 1.358688, \quad 1.023562.$
57. $x = 3.837334, \quad .931099.$
58. $x = 11.197334$, two imaginary roots.
59. $x = .465661$, two imaginary roots.
60. $x = 33.521277, \quad 21.231939.$

## PAGE 166.

The required sums of the roots are:

1. 13, 38, 117, 370, 1186.
2. −24, −24, 288, 480, −3264.
3. 10, −14, 46, −92, 256.
4. 9, 48, 145, 483, 1740.
5. −12, −15, 76, 140, −393.
6. 0, 18, 0, 50, 108.

## PAGE 166.

The remaining roots of the proposed equations are:

1. $.438447, \quad -.3027756$
2. $2.416199, \quad 2.414214.$
3. $.208712, \quad -1.381966.$
4. $-\frac{1}{2}(3.67882 \pm \sqrt{-.601169}).$
5. $-\frac{1}{2}(4.07338 \pm \sqrt{-17.7715}).$
6. $-\frac{1}{2}(4.02753 \pm \sqrt{-.662873}).$

## PAGE 175.

The solutions are given by the equations:

1. $$\left.\begin{array}{l} x^2 + 2y^2 - 34 = 0 \\ x + y - 7 = 0 \end{array}\right\}, \quad \left.\begin{array}{l} x^2 + 2y^2 - 34 = 0 \\ x - 2y = 0 \end{array}\right\}.$$
$$\left.\begin{array}{l} x - 1 = 0 \\ x + y - 7 = 0 \end{array}\right\}, \quad \left.\begin{array}{l} x - 1 = 0 \\ x - 2y = 0 \end{array}\right\}.$$

2. $y^2 - 2y = 0$ and $(y-1)x + y = 0.$

3. $$\left.\begin{array}{l} y - 1 = 0 \\ x^2 - (4y+1)x + y^2 + 5y = 0 \end{array}\right\},$$
$$\left.\begin{array}{l} x - 1 = 0 \\ x^2 - (4y+1)x + y^2 + 5y = 0 \end{array}\right\}.$$

4. $y^2 - 1 = 0$ and $(y-1)x + 2y = 0.$

5. $$\left.\begin{array}{l} (3y-10)x + y^2 + 6y = 0 \\ y^4 + 12y^3 + 87y^2 - 200y + 100 = 0 \end{array}\right\}.$$